Review of Economic Analysis 5 (2013) 177–189 1973-3909/2013177

John von Neumann between Physics and Economics: A methodological note

LUCA LAMBERTINI[*†]
University of Bologna

A methodological discussion is proposed, aiming at illustrating an analogy between game theory in particular (and mathematical economics in general) and quantum mechanics. This analogy relies on the equivalence of the two fundamental operators employed in the two fields, namely, the expected value in economics and the density matrix in quantum physics. I conjecture that this coincidence can be traced back to the contributions of von Neumann in both disciplines.

Keywords: expected value, density matrix, uncertainty, quantum games

JEL Classifications: B25, B41, C70

1 Introduction

Over the last twenty years, a growing amount of attention has been devoted to the history of game theory. Among other reasons, this interest can largely be justified on the basis of the Nobel prize to John Nash, John Harsanyi and Reinhard Selten in 1994, to Robert Aumann and Thomas Schelling in 2005 and to Leonid Hurwicz, Eric Maskin and Roger Myerson in 2007 (for mechanism design).[1]

However, the literature dealing with the history of game theory mainly adopts an *inner* perspective, i.e., an angle that allows us to reconstruct the developments of this sub-discipline under the general headings of economics. My aim is different, to the extent that I intend to propose an interpretation of the formal relationships between game theory (and economics) and the hard sciences. Strictly speaking, this view is not new, as the idea that von Neumann's interest in mathematics, logic and quantum mechanics is critical to our understanding of the genesis of

*I would like to thank Jurek Konieczny (Editor), an anonymous referee, Corrado Benassi, Ennio Cavazzuti, George Leitmann, Massimo Marinacci, Stephen Martin, Manuela Mosca and Arsen Palestini for insightful comments and discussion. The usual disclaimer applies.

†Corresponding author: luca.lambertini@unibo.it

[1] On the history of game theory, see Aumann (1985, 1987, 1999), Weintraub (1992), Leonard (1994, 1995) and Myerson (1999), *inter alia*. Leonard (2010) delves into the details of the joint venture between von Neumann and Morgenstern (1944) as well as the intellectual environment that provided the background to their *Theory of Games and Economic Behavior*.

game theory can be found in Mirowski (1992).[2] There, however, the links between quantum mechanics and mathematical economics in general (or game theory in particular) are not formally illustrated. Here, my aim is to complement the historical perspective on this issue with a methodological view on it.

A largely accepted view maintains that the foundations of both game theory and, in general, contemporary mathematical economics are to be found in *The Theory of Games and Economic Behavior* (von Neumann and Morgenstern, 1944).[3] From that work onwards, we observe the widespread use of the operator known as *expected value* in economics. Traditionally, and correctly, the roots of such operator are traced back to statistics (see Fishburn and Wakker, 1995, and the references therein). However, this view does not grasp an interesting aspect in common between mathematical economics and physics, in particular quantum physics, that is, the fact that both theories transmit a largely analogous message concerning our knowledge (or representation) of real world phenomena. Our *ex ante* knowledge (i.e., our forecasting capability) is defined in the same probabilistic terms, independently of whether we examine the behaviour of subatomic particles or the behaviour of firms and consumers.

What I want to show in the remainder is that this coincidence (pointing ultimately to the idea that our knowledge is modelled in the same way irrespective of the specific field[4]), has been largely shaped through the activity of a single researcher, John von Neumann, who, over a dozen years (1932-44) has ‘fixed’ the basic ideas in both disciplines by giving them the same methodology, thus making them largely *isomorphic* to each other. The key to this interpretation is the operator labelled as *expected value* by economists and *density matrix* by physicists.

In the first edition (in German) of *Mathematical Foundations of Quantum Mechanics*, von Neumann (1932) adopts the term *Erwartungswert*, which in English is *expectation value*. This also appears in von Neumann and Morgenstern (1944), while in the following economic literature it usually becomes *expected value*. In physics, *expectation value* is used to indicate the elementary notion of the mathematical expectation of a random variable, while *density matrix* indicates, rather loosely speaking, the random outcome of an experiment.

In the following section, I offer a succinct view of the key aspects suggesting an analogy between the two theories (and the way they build up their predictive power), through a well known parable in quantum physics.

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[2]See also Karsten (1990) and Walker (1991). A wider view on the genesis of mathematical economics following the arising of neoclassical theory in the 1870s, parallel to and patterned on classical mechanics, can be found in Mirowski (2002).

[3]In von Neumann and Morgenstern (1944), the concept of stable set for cooperative games complements the minimax theorem (von Neumann, 1928) for noncooperative games. However, the latter is applicable only in zero sum games. Nash (1950) introduced an equilibrium concept for non-zero sum noncooperative games, and the bargaining solution for cooperative games (Nash, 1951).

[4]On this issue, see Penrose (1989).

2 The analogy between mathematical economics and quantum physics: a sketch

Quantum physics (and its proper subset, quantum mechanics) can approximately be considered (not all physicists agree on this matter) as a generalisation of Newtonian physics (and mechanics). In particular, when the phenomena being investigated involve (i) speeds which are considerably lower than the speed of light in vacuum; (ii) sufficiently low gravity; and (iii) objects which are neither too small nor too big, then one can proceed according to the standard Newtonian model. Yet, when it comes either to the very basic particles (e.g., sub-atomic particles like electrons and sub-sub-atomic components like quarks), or to the behaviour of objects characterised by an extremely high mass and therefore also gravitational force (e.g., black holes, quasars and pulsars), quantal features become so relevant that they cannot be disregarded.

As an illustrative example, consider the story traditionally known as *Schrödinger's paradox* or *Schrödinger's cat-in-the-box*, which I am about to tell according to a *vulgata* commonly adopted in the current literature in the field.[5]

This paradox refers to an experiment, where a cat is locked in a box, and tied in front of a handgun, which is loaded.[6] The box is opaque to both light and sound, so that the physicist must open it to observe the cat's conditions. The trigger is linked to a sensor, located outside the box, in front of a bulb. Assume the physicist can switch on the bulb, producing a single quantum of light (a photon) at a time. The sensor measures the spin of the photon. In the remainder, I will assume conventionally the following. If the photon rotates rightwards (i.e., the spin is positive), then the sensor pulls the rope (and the trigger), and the handgun shoots the cat dead. If instead the photon rotates leftwards (i.e., the spin is negative), then the sensor does not pull the rope (and the trigger), so that the cat survives.

The paradox takes place in the physicist's mind before the experiment is carried out, or, equivalently, before he opens the box to see whether the cat is still alive or not. The physicist knows that, *ex post*, the cat is going to be either dead or alive. However, *ex ante*, the two states of the system-cat (i.e., *alive* and *dead*) as well as the two states of the world (that can be labelled as *the handgun shot because the spin was positive*, and *the handgun didn't shoot because the spin was negative*) coexist in the experimenter's mind. This paradox persists until he opens the

[5]Erwin Scrhödinger, together with Bohr, Einstein, Dirac, Heisenberg and Pauli, is one of the founding fathers of quantum physics. The remainder of this section borrows from Hawking and Penrose (1996) and Penrose (1997). For further (and much more technical) readings, see Sakurai (1985), Schwabl (1992), and Gasiorowicz (1996), *inter alia.*

[6]This can be viewed as a *Gedankenexperiment* (thought experiment), in that we must not necessarily lock a cat into a box to verify the validity of what follows. In general, a *Gedankenexperiment* "is consistent with the known laws of physics, even though it may not be technically feasible. So, measuring the acceleration due to gravity on the surface of the sun is a *Gedankenexperiment*, whereas measuring the Doppler shift of sunlight as seen from a space ship moving with twice the velocity of light is nonsense" (Gasiorowicz, 1996, p. 21, fn. 14). In the present case, replace *technically feasible* with *politically correct.*

box to observe the cat's health conditions.

I will formalise the experiment by introducing the following:

Definition 1 $\psi \equiv \{ca, cd\}$ is the wave function, or the state vector of the system (the cat).

In Definition 1, intuitively, ca = "the cat is alive"; cd = "the cat is dead". Moreover,

Definition 2 $\vartheta \equiv \{hs, hds\}$ is the vector of the states of the world.

In Definition 2, intuitively, hs = "the handgun shot (because the spin was positive)"; hds = "the handgun did not shoot (because the spin was negative)".

Finally, we have

Definition 3 $\varphi \equiv \{ca, cd\}$ is the state vector of the physicist carrying out the experiment.

The reason why φ seems to coincide with the wave function ψ will become clear in the remainder. The *ex ante* overlapping between the states of the system, is accounted for by the *total state vector of the system*:

$$|\psi_{tot}\rangle = \omega|ca\rangle + \rho|cd\rangle \tag{1}$$

where both ω and ρ belong to $\mathbb{C}$ and $|\cdot\rangle$ is called *ket*. Its complement $\langle\cdot|$ is called *bra*, and $\langle\cdot|\cdot\rangle$ is called *Dirac's brackets*, after P.A.M. Dirac who introduced this terminology in 1930 (see Dirac, 1930, 1958[4]; see also Dirac, 1925).

As soon as the experimenter opens the box to observe the state of the system, the wave function ψ *collapses*, that is, there takes place the so-called *reduction of the state vector* $|\psi_{tot}\rangle$. Hence, the experimenter observes either ca or cd, but surely not both states at the same time. On the contrary, as long as the experimenter does not open the box, the two state *coexist in a quantal sense*.[7]

For reasons that I do not dwell upon here, ω and ρ are complex numbers.[8] In particular, what I am interested in, is the information that the modulus of both ω and ρ is equal to $1/\sqrt{2}$. To the square, this yields $1/2$, which is the probability that the photon have either a positive or a negative spin. This, as we shall see in the remainder, has relevant bearings upon my aim of showing the existence of an isomorphism between quantum physics and mathematical economics (game theory in particular).

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[7]Physicists have lively arguments concerning the alternative interpretation of such a statement, i.e., whether states ca and cd coexist in parallel universes or in the experimenter's mind. As far as the present paper is concerned this distinction, although intriguing, can be disregarded.

[8]See the appendix. For further discussion, see, *inter alia*, Sakurai (1985), Schwabl (1992) and Gasiorowicz (1996).

The total state vector of the experimenter can be constructed to look like that of the system (but recall that the two vectors *do not coincide*):

$$\langle \varphi_{tot}| = \omega \langle ca| + \rho \langle cd| \ , \ \omega = \rho = 1/\sqrt{2} \ . \tag{2}$$

In probabilistic terms, the experimenter's *ex ante* knowledge about the states that the system can take *ex post* is summarised by the so-called *density matrix*:

$$D = \frac{1}{2}|ca\rangle\langle ca| + \frac{1}{2}|cd\rangle\langle cd| \ , \tag{3}$$

which is defined as the scalar product of the two total state vectors, i.e., $|\psi_{tot}\rangle \cdot \langle \varphi_{tot}|$. The introduction of the density operator can be traced back to Dirac (1930, 1958[4]) and reappears in von Neumann (1932 [1955]). This indicates that the isomorphism between (i) the theory of expected utility, game theory and contemporary mathematical economics in general, on one side; and (ii) quantum mechanics on the other side, can hardly be considered as accidental.

In *Theory of Games and Economic Behavior*, von Neumann makes use of a toolkit borrowed from Dirac's formalisation of quantum mechanics,[9] However, von Neumann does not bother to make it explicitly known either to economists or to physicists.

Expression (3) can be rewritten equivalently as follows:

$$D = \frac{1}{2}\langle ca|ca\rangle + \frac{1}{2}\langle cd|cd\rangle \ , \tag{4}$$

from which there emerges a clear analogy with what economists would call *expected value*. Indeed, $\omega^2 = \rho^2 = 1/2$ are the probabilities that the spin of the photon be either positive or negative. Therefore, the density matrix corresponds to the expected value of the experiment, which an economist would write:

$$E(\mathit{experiment}) = \frac{1}{2}(ca \mid ca) + \frac{1}{2}(cd \mid cd) \ . \tag{5}$$

Notice that the above reads the same in both disciplines, independently of whether we write it as in (4) or as in (5), i.e., "with probability 1/2 I will observe a dead cat because the cat is in fact dead (or, because the handgun shot, as the spin was positive), and with the same probability I will observe that the cat is still alive (because the handgun did not shoot, as the spin was negative)". Observe that the meaning of symbol | in brackets is exactly the same in both cases.

[9] von Neumann himself motivates his interest in quantum physics on the basis of his dissatisfaction with Dirac's formalisation of the theory (Dirac, 1930, 1958[4]), in particular with the δ function (see the author's preface in von Neumann, 1932 [1955]).

The above describes the procedure usually adopted in economics to solve games in mixed strategies.[10]

At this stage, one could rewrite (5) in terms of the vector of the states of the world, ϑ :

$$E(esp) = \frac{1}{2}(ca \mid hds) + \frac{1}{2}(cd \mid hs) . \tag{6}$$

Now I would like to stress that, when economists build up the expected value of an agent's payoff, they face exactly the same kind of paradox which I have illustrated above concerning the reduction of the state vector in the *Schrödinger's cat* example. Hence, also in economics the issue is the transition from the *ex ante* (quantal) prevision to the *ex post* (Newtonian) observation. In both settings, the limit to our knowledge is inherently given by the **indeterminacy**[11] associated to (i) the measurement in physics and (ii) the (random) transition from a mixed strategy to a particular pure strategy in game theory.[12]

The analogy with well known situations in game theory is quite immediate. Consider the following 2×2 noncooperative game in normal form: each of the two players (A and B) has two pure strategies, $\{u, d\}$ for A and $\{l, r\}$ for B. The cells identified by the intersections of rows and columns identify four possible outcomes, giving rise to individual payoffs $\pi_i(j, k)$ for player $i = A, B$, with $j = u, d$, $u = up$; $d = down$, and $k = l, r$, $l = left$; $r = right$. The game takes place under complete, symmetric but imperfect information, implying that both players know the structure of the game and have the same amount of information, but the simultaneity of moves makes it impossible for either player to observe the rival's move before choosing his/her own strategy.

		B	
		l	r
A	u	$\pi_A(u,l), \pi_B(u,l)$	$\pi_A(u,r), \pi_B(u,r)$
	d	$\pi_A(d,l), \pi_B(d,l)$	$\pi_A(d,r), \pi_B(d,r)$

Matrix 1

[10]The same discussions taking place amongst economists concerning the interpretation of a Nash equilibrium in mixed strategies as opposed to the pure-strategy equilibrium, also occur amongst physicists as to the credibility of the quantistic description of the world, as opposed to the Newtonian description. The latter discussion can be traced back to the famous dispute between Niels Bohr and Albert Einstein, who, while being one the fathers (although less than voluntarily) of quantum theory, claimed that "God doesn't toss any dies". For a contemporary revisitation of the dispute, see Hawking and Penrose (1996), with Hawking in Bohr's role and Penrose in Einstein's.

[11]This is the so-called Heisenberg's principle, which can be given a *vulgata* through the following example. We cannot know at the same time how fast an electron is moving, *and* where it is. The knowledge we can produce confines to the wave function, yielding the probability that the electron be within a certain interval, given its speed.

[12]Observe that the interpretation I am proposing here does not modify the established wisdom about the source and evolution of the theory of expected utility (see, *inter alia*, Fishburn and Wakker, 1995).

In general, the game represented in matrix 1 has at least an equilibrium in mixed strategies (and possibly several in pure strategies). Define as α the probability that player A chooses strategy a, and β the probability that player B chooses s. Obviously, the probabilities that b and d are chosen are, respectively, $1-\alpha$ and $1-\beta$. We can write the expected value of the payoff accruing to player A :

$$\begin{aligned} E(\pi_A) &= \alpha\beta\pi_A(u,l) + \alpha(1-\beta)\pi_A(u,r) + \\ &\quad (1-\alpha)\beta\pi_A(d,l) + (1-\alpha)(1-\beta)\pi_A(d,r) \ , \end{aligned} \tag{7}$$

which A must maximise w.r.t. α. We can construct $E(\pi_A)$ likewise. In the physicists' jargon, expression (7) could be defined as the *density matrix of the game* for player A, and could be rearranged as follows:

$$\begin{aligned} D(\pi_A) &= \alpha\beta\langle\pi_A(u,l)|l\rangle + \alpha(1-\beta)\langle\pi_A(u,r)|r\rangle + \\ &\quad (1-\alpha)\beta\langle\pi_A(d,l)|l\rangle + (1-\alpha)(1-\beta)\langle\pi_A(d,r)|r\rangle \ . \end{aligned} \tag{8}$$

As is well known, the matter is of the utmost relevance in games in which either there exists no Nash equilibrium in pure strategies, or there exist several ones. Matching pennies is an example of the first type (information is again complete, symmetric but imperfect):

		B	
		h	t
A	h	$1,-1$	$-1,1$
	t	$-1,1$	$1,-1$

Matrix 2

In this game, each player has two pure strategies, h (*heads*) and t (*tails*). Players are flipping two identical and 'perfect' coins, so that each coin has the same probability to generate two mutually exclusive faces when it comes to rest (or is caught and shown to both players). The examination of the matrix reveals that along the main diagonal (where both agents 'play symmetrically', which amounts to saying that both coins show the same face-up side), A pays, say, one euro to B, while the opposite happens along the secondary diagonal, in correspondence of both asymmetric outcomes. Hence, either A or B has - in turn - a strict incentive to deviate unilaterally from each strategy combination, and consequently the game has no Nash equilibrium in pure strategies. It is also worth noting that this is a zero-sum game, to clarify that the same argument applies as well to the sub-class of constant sum games that one would commonly solve via von Neumann's minimax (or maximin) criterion.

However, the equilibrium does exist in mixed strategies, and can be easily computed to ascertain that equilibrium probabilities are all equal to 1/2. This entails that the optimal mixed strategy of player $i = A, B$ is $\sigma_i^* = (h + t)/2$, delivering an expected payoff equal to zero. Yet, strategy $\sigma_i^* = (h + t)/2$ is not what players actually do in the game, and obviously neither of them is going to receive a null payoff: our *a priori* description of their behaviour and the resulting consequences does not coincide with *ex post* observation, as probabilities are necessarily going to collapse to 0 and 1 in correspondence of one of the four possible outcomes at the intersection of pure strategies.

One could raise the objection that, in a game, probabilities are chosen by players - i.e., uncertainty is a consequence of the players' trategic behaviour - while in physics they are given by Nature - and therefore uncertainty has an epistemic nature. As a counter-objection, one could put forward the model of the market for lemons (Akerlof, 1970) or the games illustrated in Bassan, Scarsini and Zamir (1997), or else point to the games with incomplete information (Harsanyi, 1967), where the states of the world are mutually exclusive and realise with probabilities dictated by Nature.

As an illustrative example of the latter class of games, one could examine the behaviour of a firm (say, for simplicity, a monopolist) in a market where the demand function is stochastic (see Leland, 1972; Klemperer and Meyer, 1986, 1989, *inter alia*). Suppose demand can take any value ϑ_i, $i = 1, 2, ...n$, with $\vartheta_i > \vartheta_{i-1}$. Define as

- $p_i(\vartheta_i)$ the probability of ϑ_i, with $\sum_{i=1}^{n} p_i(\vartheta_i) = 1$;
- $\pi_i(\vartheta_i, q_i)$ the profit accruing to the firm in state i .

The states of the world $\{\vartheta_i\}$ may refer to the political situation, the oil crisis, and so on and so forth. What matters is that they relate the profits of the firm to exogenous environmental features controlled by Nature. *Ex ante*, the firm chooses the output level in order to maximise the *expected value* of profits, i.e., $E(\pi) = \sum_{i=1}^{n} p_i(\vartheta_i) \cdot \pi_i(\vartheta_i, q_i)$. This could be translated in the jargon of physics by saying that the firm aims at maximising the density matrix of profits. *Ex post*, the equilibrium profits accruing to the firm will depend upon the precise realisation of ϑ. That is, the value of π becomes known as soon as the reduction of the state vector takes place.

There are economic problems where the economic agent's *ex ante* knowledge is sufficient to reveal with certainty how the situation will look like *ex post*. To illustrate this issue, consider the following simplified version of Akerlof's (1970) lemons market, where lemons are low-quality second-hand cars. A customer, interested in buying a second-hand car, faces a population of sellers, each offering a car whose quality can be either $q = H$ or $q = L$ ($H > L$), with probability $p(H) = p(L) = 1/2$. The metaphor usually adopted to tell this story is that Nature extracts randomly one seller from the population of sellers, half of which have good quality cars. Therefore, *a priori*, the potential buyer can compute the expected quality as $E(q) = (H + L)/2$. This is also the price the buyer is willing to pay. In the symbology adopted in physics, such expression would write as the following density matrix of quality:

$$D(q) = \frac{1}{2}\langle H|H\rangle + \frac{1}{2}\langle L|L\rangle \ . \tag{9}$$

This, *ex ante*, entails the overlapping of qualities (or states) in a quantal sense. In the present case, however, the collapse of such overlapping takes place before the transaction is carried out. To see this, examine the sellers' perspective. Each seller knows the quality of his own car. Hence, any car which is being offered on the market, given the buyer's evaluation $E(q) = (H + L)/2$, must necessarily be a poor quality one. Therefore market fails because the buyer knows that he could only buy a lemon at too high a price.

The analogy between quantum mechanics and mathematical economics outlined in the foregoing discussion is now evolving into a full isomorphism between quantum theory and game theory, in particular.[13] This can be ascertained by examining the parallel flourishing of quantal algorithms for the solution of games by game theorists (McKelvey and Palfrey, 1995, 1996, 1998) and theoretical physicists (Meyer, 1999; Eisert, Wilkens and Lewenstein, 1999; Benjamin and Hayden, 2001; Piotrowski and Sładowski, 2002, 2003a,b, 2004; Lo and Kiang, 2005; Zhou *et al.* 2005, to mention only a few).[14] This literature applies statistical models of quantal choice to noncooperative games, where each player chooses strategies based upon relative expected utility, and assume that the other players do likewise. In such a setting, a quantal response equilibrium can be defined as a fixed point, in correspondence of which players do not necessarily always select their best replies to the rival's strategies. This latest consequence of von Neumann's work belongs to the chronicle of the current evolution of game theory, rather than to the recapitulation of its history.

3 Concluding remarks

The foregoing exposition is meant to point out that our current approach to the formal analysis of issues belonging to seemingly unrelated fields of research is indeed driven by a unique tool stemming from the work of John von Neumann in physics first, and then in economics.

Besides highlighting the contribution of a single mind to two different disciplines, the discussion carried out in this paper could additionally, and more ambitiously, suggest that our knowledge of the outer world, be that the economic environment or the world of particles and forces, is governed by one single frame characterising our approach to the investigation of complex systems (cf. Penrose, 1989).

[13] In a similar, although not equivalent perspective, one may say that observation is inherently *entangled* with the system being observed: "In terms of quantum theory, investigations of microeconomic processes are interwoven with the macroeconomic frame of reference of the observer. The very act of observation is to be considered as an integral part of the observed system" (Karsten, 1990, p. 385).

[14] An illuminating view on this debate and, more specifically, on quantal vs classical correlation in games can be found in Brandenburger (2010).

Appendix: Dirac's notation

Dirac (1930, 1958[4]) introduced the following notation that applies to finite dimensional vector spaces as well as to Hilbert spaces.

Associate to the wave function ψ a state vector $|\psi\rangle$, called *ket*, and define the quantity $\langle\varphi|$ as *bra*.[15] Then,

$$\int dx\varphi^*\psi = \langle\varphi|\psi\rangle . \tag{a1}$$

The integral involving an operator W can be written as

$$\int dx\varphi^* W\psi = \langle\varphi|W\psi\rangle = \langle\varphi| W |\psi\rangle . \tag{a2}$$

If a (complex) number ω is considered, then it can be taken out of brackets, with:

$$\langle\varphi|\omega\psi\rangle = \omega \langle\varphi|\psi\rangle \tag{a3}$$

and

$$\langle\omega\varphi|\psi\rangle = \omega^* \langle\varphi|\psi\rangle \tag{a4}$$

Moreover, the following properties hold:

$$1 |\psi\rangle = |\psi\rangle \tag{a5}$$

$$|\psi\rangle + 0 = |\psi\rangle \tag{a6}$$

$$|\psi\rangle + |\varphi\rangle = |\varphi\rangle + |\psi\rangle \text{ (commutative property)} \tag{a7}$$

$$(|\psi\rangle + |\varphi\rangle) + |\omega\rangle = |\psi\rangle + (|\varphi\rangle + |\omega\rangle) \text{ (associative property)} \tag{a8}$$

$$\omega (\rho |\psi\rangle) = (\omega\rho) |\psi\rangle \text{ (associative property)} \tag{a9}$$

$$(\omega + \rho) |\psi\rangle = \omega |\psi\rangle + \rho |\psi\rangle \text{ (distributive property)} \tag{a10}$$

$$\omega (|\psi\rangle + |\varphi\rangle) = \omega |\psi\rangle + \omega |\varphi\rangle \text{ (distributive property)} \tag{a11}$$

The density operator is

$$D = |\psi\rangle \langle\psi| ; \tag{a12}$$

now, since $|\psi\rangle = \sum_i \omega_i |u_i\rangle$, we have

$$D = |\psi\rangle \langle\psi| = \sum_i (\omega_i)^2 |u_i\rangle \langle u_i| \tag{a13}$$

or, more generally,

$$D = |\psi\rangle \langle\psi| = \sum_i \omega_i\omega_j |u_i\rangle \left\langle u_j\right| . \tag{a14}$$

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[15] φ^* is the complex coniugate wave function. See Gasiorowicz (1996).

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