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# Global univalence when mappings are not necessarily continuous*

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This paper proposes a method of establishing the global univalence of a mapping without the assumption of continuity and the absence of points of inflection. When the functions are not continuous and the points of inflections are present, the use of a Jacobian to establish univalence presents some difficulties. The method of establishing univalency, presented in this paper, in turn generalizes the theorems on the uniqueness of competitive equilibrium and factor price equalization.

*Key words:* Continuity; Univalence; Dominance; Equilibrium
*JEL classifications:* C60; D50

## 1. Introduction

The study of univalent mappings has been considered in economics in the context of establishing the uniqueness of competitive equilibrium, the factor-price equalization theorem and the generalized Leontief system. Traditionally, the univalence of a mapping is ensured by examining the Jacobian matrix which requires the mapping to be differentiable, e.g. the Jacobian is to be a P- (or NP-) matrix [Nikaido (1968)] or quasi-dominant diagonal matrix [McKenzie (1959)] or it should satisfy related restrictions [Mas-Colell (1970, 1979, 1985), Pearce and Wise (1973, 1974), Kehoe (1985)]. In an interesting development of the literature on univalency, Mas-Colell and Kehoe [see Mas-Colell (1985)] developed the 'index theory' using a bordered Jacobian. However, like others' works mentioned above, their approach is also

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restricted to the analysis of smooth economies with differentiable excess demand functions. The differentiability of a mapping is indeed a stringent assumption when we recall that in order to prove the existence of a competitive equilibrium we take pains to avoid the assumption of continuity of excess demand correspondence [Arrow and Hahn (1971)]. Considering the initial endowment of an individual at the boundary of the consumption set, it is easy to construct a simple example to see the possibility of discontinuity in the (excess) demand correspondence.[1] In the context of the factor-price equalization theorem, the unit cost functions are not necessarily differentiable everywhere in the domain (e.g., when the unit isoquant is the convex combination of a set of discrete points). Hence, the study of univalency without the assumptions of continuity and differentiability is of considerable significance in economics. In the absence of continuity, the Jacobian techniques do not work in examining univalency.[2] Hence, in the absence of continuity we need an alternative approach to establish the univalency of a mapping. Furthermore, in all results on global univalence, the conditions imposed on the Jacobian imply that the Jacobian and all its principal minors are bounded away from zero. Vanishing of the Jacobian does not necessarily imply degenerate mapping (absence of local homeomorphism). It may happen at points of inflection, which is admissible under univalent mapping.[3] Apart from inflection, there is another interesting case where vanishing Jacobians may be associated with univalent mappings. For a better intuitive understanding, we shall discuss this case in section 5.

## 2. The preliminaries

We shall be working in the real space $\mathbb{R}^n$; the representation of dimensionality $n$ of the space depends on the economic models. For example, $x \in \mathbb{R}^n$ represents factor prices in the context of the factor-price equalization theorem and commodity prices in the context of competitive equilibrium.

[1]For the definition of consumption set, see Arrow and Hahn (1971). In economic literature, discontinuous functions are typically associated with correspondences and, in particular, upper semi-continuous correspondences. One important aspect of our approach is that it can deal with such complexities.

[2]For the sake of a simple illustration, consider a function $y=f(x)$ such that

$$y=60+7x \quad \text{for} \quad 0\leq x<20,$$
$$y=\quad 4x \quad \text{for} \quad 20\leq x<\infty.$$

The function $f(x)$ has positive derivatives almost *everywhere* inside the domain (hence, trivially satisfies the P-matrix criterion) except at $x=20$ where the discontinuity occurs. Although the P-matrix criterion is satisfied almost everywhere, the function is not univalent (e.g., for $y=172$, there are two solutions $x=16$ and $x=43$).

[3]The points of inflection alone do not create much problem. In fact, it is trivial to perturb away inflection points of univalent mappings. Furthermore, in certain cases, the study of univalency involving nondifferentiable mappings could be handled by adopting the technique of nonsmooth analysis [see Hiriar-Urruty (1981, 1982)].

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Following standard practice, for $x$ and $x'$ in $\mathbb{R}^n$ the notation $x' \geqq x$ implies $x'_i \geq x_i$ for all $i$, $i \doteqdot 1, \ldots, n$. The notation $x' \geq x$ implies $x'_i \geq x_i$ for all $i$ and $x'_i > x_i$ for some $i$. Let $\Omega$ denote the nonnegative orthant $\mathbb{R}^n$. We are interested in the following set of equations:

$$y_i = f_i(x_i, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{1}$$

In section 3, we assume $x, y \in \Omega$, where we discuss the case of nondecreasing functions. However, the main theorem in this section has been stated in a more general context. In terms of economic interpretation, this case is related to the factor-price equalization theorem where the factor prices and the commodity prices are nonnegative. On the other hand in section 4, both the domain and the range of the mapping $y = f(x)$ are assumed to be unrestricted. Again in terms of economic interpretation, this section is related to the study of uniqueness of competitive equilibrium. Excess demands may be either negative or positive. Similarly, prices may be negative if we do not allow for free disposability. If we assume that commodities are freely disposable, then the prices $(x)$ are nonnegative. The theorems proved in section 4 may also be extended to this case using logarithmic transformation of variables. Note, (i) $\log y$, $\log x \in \mathbb{R}^n$ where $x, y \in \Omega$, (ii) if $y$ is an increasing function in $x$, it will also be an increasing function in $\log x$.

*Definition 1.* Let $A = [a_{ij}]$ be a real $n \times n$ matrix. $A$ is said to be

(1.1) *Dominant diagonal* if $|a_{jj}| > \sum_{i \neq j} |a_{ij}|$ for each $j$.
(1.2) *Generalized dominant diagonal* if there exists a number, $d_j > 0$, $j = 1, 2, \ldots, n$ such that $d_j |a_{jj}| > \sum_{i \neq j} d_i |a_{ij}|$.
(1.3) *Quasi-dominant diagonal* if there exists $d_j > 0$, such that $d_j |a_{jj}| \geq \sum_{i \neq j} d_i |a_{ij}|$ $(j = 1, \ldots, n)$, and when $a_{ij} = 0$ (given $j \in J$ and $i \neq j$ for some set of indices $J$), the strict inequality holds for some $j \in J$.
(1.4) *P-matrix* if all principle minors are positive.
(1.5) *NP-matrix* if all principle minors of odd orders are negative and those of even orders are positive.
(1.6) *Weak gross substitute* (WGS) if $a_{jj} \leq 0$ and $a_{ij} \geq 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$.

McKenzie (1959, Corollary of Theorem 4) has established that any square matrix which is quasi-dominant diagonal is also a generalized dominant diagonal matrix. The importance of the various dominant diagonal matrices lies in their applications in a number of economic problems. A matrix having positive (negative) dominant diagonal is a P- (NP-) matrix. In general a P- or NP-matrix is not a quasi-dominant (or generalized dominant) diagonal matrix. However, if a matrix has a specific sign-pattern, e.g. nonnegative (or WGS) sign pattern, then a P- (or NP-) matrix is a quasi-dominant diagonal matrix. In economics, quite often we are concerned with a Jacobian matrix whose sign structure is known. Two such frequently discussed signed-

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matrices are (i) nonnegative matrices and (ii) WGS matrices. For example, the Jacobian of the unit cost functions or the Jacobian of the input–output functions is a nonnegative matrix. In general equilibrium theory, the case of weak gross substitutes yields a Jacobian of the excess demand functions which is a WGS matrix. However, as we have remarked earlier, the assumption that the underlying functions are continuous or the principal minors are bounded away from zero, imposes some severe restrictions. We shall now explain why the univalency of a mapping may be established without such restrictions by presenting an alternative approach.

Let us first investigate some implications of the dominant diagonal property which would provide an insight for the alternative approach. Let a set of functions (1) be differentiable and increasing with respect to all its arguments, i.e., $f_{ij}=\delta y_i/\delta x_j>0$ for all $i$ and $j$. The Jacobian is obviously a positive matrix. The above equations may alternatively be written as

$$y_i=f_i(\{(x_1+x_i)-x_i\},\ldots,x_i,\ldots,\{(x_n+x_i)-x_i\})=P_i(t_{1i},..t_{ii},..t_{ni}),$$

where $t_{ji}=x_j+x_i$, $i\neq j$ and $t_{ii}=x_i$. Note, $T=(t_{ji})\in\Omega$ for $x\in\Omega$. Clearly, $p_{ij}=(\delta P_i/\delta t_{ji})=(\delta f_i/\delta x_j)>0$, $i\neq j$; and $p_{ii}=(\delta P_i/\delta t_{ii})=[(\delta f_i/\delta x_i)-\sum(\delta f_i/\delta x_j)]>0$ if the Jacobian $[f_{ij}]$ is a positive dominant diagonal matrix. It is now obvious that the positive dominant diagonal property of the Jacobian of a mapping $y=f(x)$, where $x,y\in\Omega$, implies that the transformed functions $y_i=P_i(t_{1i},\ldots,t_{ni})$, $i=1,2,\ldots,n$, have positive partial derivatives, $(\delta y_i/\delta t_{ji})>0$, for all $i$ and $j$.

This observation on the positive dominant diagonal property suggests to our intuition that for a certain class of mappings (e.g. when the mapping is a set of nondecreasing functions), the continuity assumption is not necessary. We may replace the $\delta y_i/\delta t_{ji}\geqq 0$ with the property that $y_i$ is a nondecreasing function in $t_{ji}$ $(j=1,\ldots,n)$ even allowing for 'jumps'.

## 3. Univalency: The case of nondecreasing mappings

A mapping $y=f(x)$ is said to be *univalent* if for any assigned $y$-vector the solution vector $x$ is unique. The function $f_i$ is *increasing* with respect to $x_j$ if $f_i(x'_1,\ldots,x'_n)>f_i(x_1,\ldots,x_n)$ for $x'_j>x_j$ and $x'_i=x_i$ for all $i\neq j$. The function is said to be *nondecreasing* if the strict inequality of $f_i$ is replaced by weak inequality.

For a set of increasing functions the positive dominant diagonal property of the Jacobian implies that a simultaneous increase in the $i$th variable and decrease of same magnitude in all other variables increases the value of the $i$th function. To capture the essence of this argument for nondecreasing functions which are not necessarily continuous, it is instructive to introduce a transformation function.

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*Definition 2.* Given $y_i = f_i(x_i, \ldots, x_n)$, $i = 1, 2, \ldots, n$, a function $P_i$ is defined for $t_{ji} = x_j + x_i$, $i \neq j$ and $t_{ii} = x_i$ as

$$P_i(t_{1i}, \ldots, t_{ii}, \ldots, t_{ni}) = f_i((t_{1i} - t_{ii}), \ldots, t_{ii}, \ldots, (t_{ni} - t_{ii})),$$

where $x \in \Omega$ and $T = (t_{ji}) \in \Omega$.

Although the effect of a change in $x_j$ on $P_i$ and $f_i$, where $i \neq j$, are the same, any change in $x_i$ alone affects all other arguments of the $P_i$ function. Now to examine the effect of an increase in only the $i$th argument of $P_i$ one must simultaneously increase $x_i$ and decrease all other $x_j$'s by the same amount.

*Definition 3.* Let $P_{ij}$ denote $P_i$ as a function of $t_{ji}$. The function $f_i$ is said to be

(3.1) *nonnegatively responsive* iff $P_{ii}$ is nondecreasing;
(3.2) *positively responsive* iff $P_{ii}$ is increasing;
(3.3) *sensitive* to $x_j$ iff $P_{ij}$ is increasing.

Positive responsiveness (respectively nonnegative responsiveness) requires that a simultaneous increase in $x_i$ and decrease in all other $x_j$'s increases (respectively does not change) the value of $f_i$. Sensitivity to $x_j$ says that an increase in $x_j$ results in an increase in $f_i$.

*Definition 4.* Let $y = f(x)$, $x, y \in \Omega$, be a set of nondecreasing functions. Let $P_i$ and $P_{ij}$ be defined for $i, j = 1, \ldots, n$. Then $f$ satisfies the *weakly positive dominance condition* (WPD) iff for every $i = 1, \ldots, n$, $f_i$ is nonnegatively responsive and there is a sequence $i(0), i(1), \ldots, i(k)$, where $i(0) = i$, such that for $s = 0, 1, \ldots, k-1$, $f_{i(s)}$ is sensitive to $x_{i(s+1)}$ and $f_{i(k)}$ is positively responsive.

Note that for every $i = 1, \ldots, n$, if $f_i$ is positively responsive then WPD is trivially satisfied. In other words, WPD requires that for every $i = 1, \ldots, n$ either $P_{ii}$ is increasing or $P_{ii}$ is nondecreasing and there is a sequence $i(0), i(1), \ldots, i(k)$, where $i(0) = i$, such that for $s = 0, 1, \ldots, k-1$, $P_{i(s)i(s+1)}$ and $P_{i(k)i(k)}$ are increasing.[4]

For an economic interpretation, suppose that $n$ equations given in (1) is a set of unit cost functions where $y_i$ is the price of commodity $i$ and $x_j$ is the price of factor $j$. Now the nonnegative responsiveness (i.e., nondecreasing $P_{ii}$) requires that both the goods and the factors may be ranked in such a way that an increase in $x_i$ and an equal reduction in all other $x_j$'s will not reduce $y_i$. The chain condition of WPD may be interpreted in the following way: for any $j$ in the index set, either the industry $j$ is positively responsive to the factor $j$ or there exists a sequence $j(0), j(1), \ldots, j(k)$ with $j(0) = j$ such that the industry $j(s)$ is sensitive to the price of factor $j(s+1)$ for $s = 0, 1, \ldots, k-1$

[4]The origin of a related type of mapping can be found in network theory. See Duffin (1948) where he considered continuous mapping in proving the existence and uniqueness of a solution.

and the industry $j(k)$ is positively responsive to the factor $j(k)$. In a two-good, two-factor world WPD is trivially satisfied whenever goods and factors are ranked in such a way that the price of each good is positively responsive to the price of a distinct factor.

Consider an $n \times n$ sign matrix which has a '+' (or 0) sign in the entry of $i$th row and $j$th column if $P_{ij}$ is increasing (or locally constant). The chain condition of WPD establishes the connection between the elements of the sign matirx which is constructed from the original mapping $y=f(x)$. Following is an example of a mapping which satisfies WPD but its Jacobian is neither a $P$-matrix nor a positive dominant diagonal matrix.[5]

*Example 1.* Consider the following set of functions:

$$\begin{aligned} y_1 &= (x_1-1)^3 + 2(x_1+x_2), \\ y_2 &= x_1 + x_2, \\ y_3 &= x_2 + x_3. \end{aligned} \tag{2}$$

It is easy to check that the Jacobian of (2) vanishes whenever $x_1=1$; and the Jacobian matrix of (2) is neither a P-matrix nor a positive dominant diagonal matrix for all $x \in \Omega$. Obviously, given any $(y_1, y_2, y_3) \geqq 0$, if any $x=(x_1, x_2, x_3)$ satisfies (2), then $x$ is unique. The system of eqs. (2) is a simple illustration of a univalent mapping whose univalency cannot be established by using the P-matrix or the dominant diagonal matrix technique. However, it is easy to see (2) satisfies WPD. Consider the matrix of the $P_{ij}$ functions which are nondecreasing. Use the symbol $(+)$ when $P_{ij}$ is an increasing function and (0) otherwise. The matrix $[p_{ij}]$ which corresponds to our illustration has the following sign-pattern:

$$[p_{ij}] = \begin{bmatrix} (+) & (+) & (0) \\ (+) & (0) & (0) \\ (0) & (+) & (0) \end{bmatrix}. \tag{3}$$

Note, $P_{11}$ is an increasing function in $x_1$ even at $x_1=1$ although the derivative of $(x_1-1)^3$ with respect to $x_1$ vanishes whenever $x_1=1$. From the sign-pattern of the elements in (3) it is obvious that (2) satisfies WPD.

Although, in the economic literature, we generally consider the domain of a mapping to be either $\mathbb{R}^n$ or $\Omega$, we shall state the main result of this section in a generalized form. Let $S$ be defined as a *positively comprehensive set* if

[5]The same point can be made trivially with the mapping $y=x^3$. Our example illustrates the construction of $[p_{ij}]$ and the use of the chain condition for WPD.

$S+\Omega\subset S$.[6] Now assuming that $S$ is a positively comprehensive set, we present the main result of this section:

*Theorem 1. Let $y=f(x)$, $x, y\in S$, be a set of nondecreasing functions satisfying WPD. For any assigned vector $y$ if a solution vector $x$* exists then *$x$ is unique.*

This result can be extended to semicontinuous correspondences with some suitable definitions for increasing and decreasing correspondences. Define a correspondence $F_i$ to be *increasing* in $x_j$ if $y_i'>y_i$ for *any* $y_i'\in F_i(x_1,\dots,x_{j-1},x_j',x_{j+1},\dots,x_n)$ and *any* $y_i\in F_i(x_1,\dots,x_n)$ where $x_j'>x_j$. We define a correspondence $F_i$ to be nondecreasing with respect to $x_j$ in a similar way. The reader should note that if $F_i$ is nondecreasing in its arguments, then the 'graph' of the correspondence is 'thin' (single valued $y_i$), *almost everywhere* in the domian of $x_j$. Intuitively, it should be clear, that if the graph were not thin *almost everywhere*, we would have a problem in having any reasonable theorem on univalence.

If $F_i(x)$ is a nondecreasing correspondence, then construct a set of derived funtions $y=f(x)$ choosing *any* arbitrary $y_i\in F_i$ where $F_i$ is multivalued. The set of functions $y=f(x)$ is a set of nondecreasing functions (not necessarily continuous). It is clear that if $F(x)$ satisfies WPD, so does $f(x)$. Now suppose $F(x)$ satisfies WPD and for some $y\in F(x)$, there exists $x$, $x'$ such that $y=F(x)=F(x')$. It is then possible to construct a derived function $f(x)$ – not necessarily continuous – such that $y=f(x)=f(x')$. But we know that a derived function of $F(x)$ must also satisfy WPD and by Theorem 1, $f(x)\neq f(x')$. Therefore, $F(x)$ must be univalent. A similar kind of extension to Theorem 2 in section 4 can easily be constructed.

It is obvious that if $f(x)$, $x, y\in S$, is differentiable everywhere and if its Jacobian is a nonnegative matrix which has the property of having a positive dominant diagonal, i.e., $p_{ii}=(f_{ii}-\sum_{i\neq j}f_{ij})>0$ for every $i=1,\dots,n$, then $y=f(x)$ must satisfy WPD. Example 1 shows that the converse is not true. In other words, the requirement of WPD, like $P$-matrix and quasi-dominant diagonal matrix, is weaker than the requirement of positive dominant diagonal property of the Jacobian matrix. Furthermore, as Example 1 shows, the requirement of P-matrix sometime may appear to be more demanding than necessary for univalency of a mapping.

To see the relationship between the quasi-dominant diagonal property of the Jacobian and WPD, let us define variables $x_i^*$ such that $d_i(x_i^*)=x_i$ with derivative $d_i'>0$, $i=1,\dots,n$. Apparently, $d_i'\delta x_i^*=\delta x_i$. As before one may define $t_{ji}^*=x_i^*+x_j^*$ and $t_{ii}^*=x_i^*$ that allows us to write $y_i=p_i^*(t_{1i}^*,\dots,t_{ni}^*)$. Now $p_{ij}^*=\delta P_i^*/\delta t_{ji}^*=\delta y_i/\delta x_j^*=d_j'(\delta f_i/\delta x_j)$ and $p_{ii}^*=\delta P_i^*/\delta t_{ii}^*=\delta y_i/\delta x_i^*-\sum_{j\neq i}\delta y_i/\delta x_j^*=$

[6] The comprehensive set in Levhari et al. (1975) may be regarded as a negatively comprehensive set.

$d_i'(\delta f_i/\delta x_i)-\sum_{j\neq i} d_j'(\delta f_i/\delta x_j)$. If the Jacobian of a mapping $y=f(x)$ has a positive generalized dominant diagonal with the choice of weights as $d_1', d_2', \ldots, d_n'$, then $p_{ii}^*>0$. It is clear that the positive generalized dominant diagonal property of the mapping $y=f(x)$, $x, y \in S$, implies that the transformed function $y_i=p_i^*(t_{1i}^*, \ldots, t_{ni}^*)$ has positive partial derivatives, $\delta y_i/\delta t_{ji}^*>0$ for all $i$, $j$ and $(t_{ji}^*)\in S$. Now we make the following remark.

*Remark 1.* Let $y=f(x)$, $x, y\in S$, be a set of nondecreasing and differentiable functions. If the Jacobian of $f(x)$ is a positive generalized quasi-dominant diagonal matrix with weights $d_i>0$ $(i=1,\ldots,n)$, then there exist variables $x_i^*$, where $x_i=d_i x_i^* (i=1,\ldots,n)$, such that $y=g(x^*)$ satisfies WPD. The converse is not true.

*Proof.* Since we are dealing with square matrices, by McKenzie's corollary [13], a quasi-dominant diagonal matrix is also a generalized dominant diagonal matrix. Therefore, we have to show that if the Jacobian of $y=f(x)$ is a positive generalized dominant diagonal then $y=g(x^*)$ satisfies WPD.

Define $t_{ji}^*=x_j^*+x_i^*=x_i/d_i+x_j/d_j$ and $t_{ii}^*=x_i^*=x_i/d_i$. Then $\delta P_i^*/\delta t_{ji}^*=\delta y_i/\delta x_j^*=d_j(\delta y_i/\delta x_j)\geqq 0$, and $\delta P_i^*/\delta t_{ii}^*=\delta y_i/\delta x_i^*-\sum_{j\neq i}\delta y_i/\delta x_j^*=d_i(\delta y_i/\delta x_j)-\sum_{j\neq i} d_j(\delta y_i/\delta x_j)>0$. Since the Jacobian of $y=f(x)$ satisfies the positive generalized dominant diagonal property, the above implies that $y$ expressed as a function of $x^*$ satisfies WPD trivially (i.e., we do not have to invoke the chain condition of WPD).

Example 1 shows that the converse is not true. Note, the Jacobian of the set of functions of the example remains a singular matrix for any strictly monotonic transformation of the variable $x$.

## 4. Univalency: The case of nonincreasing mappings

So far we have considered the nondecreasing functions and have established the relationship with the nonnegative Jacobian matrices used in the international trade theory and in the generalized Leontief model. In the context of competitive equilibrium analysis, the mappings we encounter, i.e., the excess demand (supply) functions, are neither nondecreasing nor nonincreasing with respect to all its arguments. In the well-known case of 'weak gross substitutes' (WGS), the excess demand functions are nonincreasing with respect to own-price and nondecreasing with respect to other prices.

Let $y_i=f_i(x_i,\ldots,x_n)$, $i=1,\ldots,n$. Throughout this section we assume that $f_i$ is nonincreasing with respect to $x_i$ and nondecreasing with respect to $x_j$, $j\neq i$. To deal with this class of mappings, it is instructive to introduce the following transformation function.

*Definition 5.* Given $y_i=f_i(x_1,\ldots,x_n)$, $i=1,2,\ldots,n$, a new function $R_i$, is

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defined for $t_{ji}=x_i-x_j$, $i \neq j$ and $t_{ii}=x_i$ as $R_i(t_{1i},\ldots,t_{ii},\ldots,t_{ni})= f_i((t_{ii}-t_{1i}),\ldots,t_{ii},\ldots,(t_{ii}-t_{ni}))$.

*Definition 6.* Let $R_{ij}$ denote $R_i$ as a function of $t_{ji}$. The function $f_i$ is said to be

(6.1) *nonpositively responsive* iff $R_{ii}$ is nonincreasing,
(6.2) *negatively responsive* iff $R_{ii}$ is decreasing,
(6.3) *sensitive* to $x_j$ iff $R_{ij}$ is decreasing.

Negative responsiveness (respectively nonpositive responsiveness) requires that a simultaneous increase in all $x_i$'s by the same amount decreases (respectively does not change) the value of $f_i$. Once again, sensitivity to $x_j$ says that an increase in $x_j$ results in an increase in $f_i$.

*Definition 7.* Let $y=f(x), x, y \in \mathbb{R}^n$, be a set of functions such that for each $i$, $f_i$ is nonincreasing in $x_i$ and nondecreasing in $x_j$ for all $j \neq i$. Let $R_i$ and $R_{ij}$ be defined for $i, j=1,\ldots,n$. Then $f$ satisfies the *weakly negative dominance condition* (WND) iff for every $i=1,\ldots,n$, $f_i$ is nonpositively responsive and there is a sequence $i(0)$, $i(1),\ldots,i(k)$, where $i(0)=i$, such that for $s=0,1,\ldots,k-1, f_{i(s)}$ is sensitive to $x_{i(s+1)}$ and $f_{i(k)}$ is negatively responsive.

Once again note that for every $i=1,\ldots,n$, if $f_i$ is negatively responsive then WND is trivially satisfied. In other words, the WND condition requires that for every $i=1,\ldots,n$ either $R_{ii}$ is decreasing or $R_{ii}$ is nonincreasing and there is a sequence $i(0)$, $i(1),\ldots,i(k)$, where $i(0)=i$, such that for $s=0,1,\ldots,k-1$, $R_{i(s)i(s+1)}$ and $R_{i(k)i(k)}$ are decreasing. In the context of WND the following result is parallel to Theorem 1.[7]

*Theorem 2. Let $y=f(x)$, $x, y \in \mathbb{R}^n$, be a set of functions such that for each $i$, $f_i$ is nonincreasing in $x_i$ and nondecreasing in $x_j$ for all $j \neq i$. Let $f$ satisfy WND. For any assigned $y$ if a solution vector $x$ exists then $x$ is unique.*

Our next theorem establishes that WND is both necessary and sufficient for global univalency of nondecreasing mappings which satisfy nonpositive responsiveness. This result in turn shows further the importance of chain relation that we have used in defining WND.

[7]Following the method of constructing $p_{ij}$'s from $p^i(\cdot)$, construct $r_{ij}$'s from $R_i$, $i=1,\ldots,n$ such that

$$\begin{aligned} 0 &= r_{11}\delta x_1 + r_{12}(\delta x_1 - \delta x_2) + \cdots + r_{1n}(\delta x_1 - \delta x_n), \\ &\vdots \\ 0 &= r_{n1}(\delta x_n - \delta x_1) + r_{n2}(\delta x_n - \delta x_2) + \cdots + r_{nn}\delta x_{n1}, \end{aligned}$$

where, by WND, $r_{ij} \leqq 0$ for $i, j=1,\ldots,n$. The rest of the proof is very similar to Theorem 1.

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*Theorem 3.* *Let $y_i = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$, be a set a functions which are nonincreasing in $x_i$ and nondecreasing in $x_j$ for $j \neq i$. Let $f_i$ $(i = 1, \ldots, n)$ be nonpositively responsive. Then $y = f(x), x, y \in \mathbb{R}^n$, is univalent if and only if $f(x)$ satisfies WND.*

Once again, following the proof of Remark 1, the observation below is immediate.

*Remark 2.* Let $y = f(x)$, $x, y \in \mathbb{R}^n$ be a set of differentiable mappings whose Jacobian is a WGS. If the Jacobian of $f(x)$ is a negative quasi-dominant diagonal matrix with weights $d_i > 0$ $(i = 1, \ldots, n)$, then there exist variables $x_i^*$, where $x_i^* = d_i x_i$ $(i = 1, \ldots, n)$, such that $y = g(x^*)$, $x^* \in \mathbb{R}^n$, satisfies WND. The converse is not true.

In general a set of functions may not all be nondecreasing as required by WPD or satisfy a similar type of restriction required by WND. However, if these functions can be transformed to satisfy WPD and WND by any monotonic transformation of the variables, then univalency holds. For example let

$$x_i = \Psi_i(q_i), \quad i = 1, 2, \ldots, n, \tag{4}$$

such that $\Psi_i$ is strictly monotonic and unbounded. Substituting (4) in (1) we get

$$y_i = f_i(\Psi_1(q_1), \ldots, \Psi_n(q_n)) = F_i(q_1, \ldots, q_n), \quad i = 1, 2, \ldots, n. \tag{1'}$$

Since (1) may not be continuous, (1′) is not necessarily continuous. Now the following observation is immediate.

*Remark 3.* If (1′) satisfies WPD or WND then (1) is a univalent mapping.

As explained in section 3, Theorems 2 and 3 may be easily extended to cover the cases of 'thin' upper semicontinuous correspondences.

**5. Economic applications**

Much of the discussion of uniqueness in the economic literature was originally motivated by the theory of factor-price equalization in international trade theory. In the general model of two competitive economies with no joint production, constant returns to scale production functions, full employment and equal number of commodities and factors, commodity prices ($p_i$'s) and factor prices ($w_i$'s) in each country satisfy the relation $p = c(w)$, $p, w \in \Omega$. The factor-price equalization theorem establishes that if certain conditions

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are fulfilled by the structure of the cost functions $c(w)$, then given any commodity price vector $(p)$, the factor price vector $(w)$, if it exists, must be unique. Samuelson (1953) sought to figure out these conditions by examining the Jacobian of the mapping $c(w)$. He thought of a sufficient condition which required that for any permutation of the rows and columns of the Jacobian, all its leading principle minors are nonvanishing. With a counter-example, Gale and Nikaido (1965) showed that this proposition paid insufficient attention to the domain of the mapping. They established that if the Jacobian has the property of P-matrix, then the factor-price equalization theorem holds. We have shown earlier (by giving an example) that a mapping has a unique solution although its associated Jacobian is neither a P-matrix nor a dominant diagonal matrix. Apart from the case of inflection, there is another interesting case of univalent mapping with vanishing Jacobian. Consider unit isocost curves drawn on the factor-price space. If the curves meet tangentially (the case of factor intensity reversal), the factor-price equalization theorem does not hold. On the other hand, it is possible that the curves always intersect but at the point of intersection two curves have the same slope. In this case the vanishing Jacobian does not affect univalency. In the former case, in the neighborhood of tangency, there must exist $w$ and $w'$ such that eq. (6), presented in the next section, holds with the matrix of coefficients being singular. In this case the WPD condition is violated. In the second case, where univalency holds, the matrix of coefficients as in eq. (6) is nonsingular and the WPD condition holds. This shows that the WPD condition is more helpful in such cases than the Jacobian conditions. Now we state the factor-price equalization theorem in a suitable form.

*Proposition 1 (factor-price equalization). Let $p=c(w)$, $w,p\in\Omega$, represent the relationship between commodity prices and factor prices, i.e., a set of unit cost functions. Let $c(w)$ be a nondecreasing mapping. If $c(w)$ satisfies WPD condition, then for any assigned commodity price vector $(p)$, there exists a unique factor price vector $(w)$.*

Now, if the vectors $w'$ and $p'$ are the relative prices of factors and commodities (respectively relative to the price of the factor $j$ and the price of commodity $k$) then $c_i(w')$'s are not required to be linearly homogeneous and the Jacobian matrix of the system is not necessarily always nonnegative. In that case the use of the technique of P-matrix or dominant diagonal matrix requires knowledge beyond the sign-pattern of the Jacobian. However, utilizing Remark 3 together with Theorem 1, the following result is immediate.

*Proposition 2 (relative factor-price equalization). Let $p'=c(w')$, $w',p'\in\Omega$, represent the relationship between the relative commodity prices and the* *relative factor prices. If there exists a mapping $w' = \Psi(q)$ which satisfies (4) such that $p' = c(w)'.\Psi = F(q)$ is a nondecreasing mapping and satisfies WPD condition then for any assigned relative commodity price vector $(p')$, there exists a unique relative factor price vector $(w')$.*

The importance of the study of univalent mappings in economics has been established above all by the people who are interested in proving the uniqueness of competitive equilibrium. Let $z_i^* = z_i^*(p_1, \ldots, p_{n+1})$, $i = 1, \ldots, n+1$, be a set of excess demand functions of a competitive economy, where $z_i^*$ stands for the excess demand for commodity $i$ and $(p_1, \ldots, p_{n+1})$ represents the vector of commodity prices. It is well known that such excess demand functions are homogeneous of degree zero in prices.

A set of excess demand functions, $z_i^*(p_1, \ldots, p_{n+1})$, $i = 1, \ldots, n+1$, is said to satisfy the property of *weak gross substitute* if and only if each $z_i^*$ is a nonincreasing function in $p_i$ and nondecreasing function in $p_j$ for $j = 1, \ldots, n+1$, and $j \neq i$ and $p \in \mathbb{R}^{n+1}$. Then an economy is said to be in *competitive equilibrium* whenever for a price vector $p \in \mathbb{R}^{n+1}$, $z^*(p) = 0$. Now assuming $p_{n+1} = 1$, let us normalize the price vector. The competitive equilibrium corresponding to a set of normalized excess demand functions, $z_i(p_1, \ldots, p_n)$, $i = 1, \ldots, n$, is said to be *unique* if the set of $n$ equations, $z_i(p_1, \ldots, p_n) = 0$, has a unique solution vector. Then, following the technique in establishing Theorem 2, transform the normalized excess demand functions to

$$z_i = R_i(t_{1i}, \ldots, t_{ni}), \quad i = 1, \ldots, n,$$

where $t_{ji} = p_i - p_j$ and $t_{ii} = p_i$. Since $R_i$ satisfies the property of weak gross substitute and also since $t'_{ji}$'s are normalized, $R_{ii}$ must be a nonincreasing function. Now the result below is immediate.

*Proposition 3. If the set of excess demand functions of a competitive economy, $z_i^*(p_1, \ldots, p_{n+1})$, $i = 1, \ldots, n+1$, satisfies the property of weak gross substitute, then the normalized mapping, $z_i(p_1, \ldots, p_n)$, $i = 1, \ldots, n$, is univalent if and only if $z(p_1, \ldots, p_n)$ satisfies WND.*

## 6. Proof of the theorems

Before proving the univalency results first consider a set of nondecreasing functions $y_i = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$, which satisfies WPD. Now we describe the procedure of constructing an $n \times n$ matrix in $p_{ij}$'s corresponding to the given set of mappings. For any assigned values $y_1, y_2, \ldots, y_n$ if there exist two solution vectors $x$ and $x'$ define $t_{ii} = x_i$, $t_{ji} = (x_j + x_i)$ for $j \neq i$ and $t'_{ii} = x'_i$, $t'_{ji} = (x'_j + x'_i)$ for $j \neq i$.

First consider the case for $t_{11} \leq t'_{11}$. Then, by definition,

$$
\begin{aligned}
0&=P_1(t'_{11},\ldots,t'_{n1})-P_1(t_{11},\ldots,t_{n1})\\
&=[P_1(t'_{11},\ldots,t'_{n1})-P_1(t_{11},t'_{21},\ldots,t'_{n1})]+[P_1(t_{11},t'_{21},\ldots,t'_{n1})\\
&\quad -P_1(t_{11},t_{21},t'_{31},\ldots,t'_{n1})]+\cdots+[P_1(t_{11},\ldots,t_{(n-1)1},t'_{n1})\\
&\quad -P_1(t_{11},\ldots,t_{n1})].
\end{aligned}
\tag{5}
$$

Define, for $t'_{j1}\neq t_{j1}$,

$$p_{1j}=[P_1(t_{11},\ldots,t_{(j-1)1},t'_{j1},\ldots,t'_{n1})-P_1(t_{11},\ldots,t_{j1},t'_{(j+1)1'},\ldots,t'_{n1})].d$$

where $d=(t'_{ji}-t_{ji})^{-1}$; otherwise, for $t'_{j1}=t_{j1}$, $p_{1j}=1$. Since $t'_{j1}\geq t'_{11}\geq t_{11}$ for all $j$, $p_{1j}$'s are defined for $x,y\in S$.

Now consider the case for $t_{11}\geq t'_{11}$. Define, for $t'_{j1}\neq t_{j1}$,

$$p_{1j}=[P_1(t'_{11},\ldots,t'_{(j-1)1},t_{j1},\ldots,t_{n1})-P_1(t'_{11},\ldots,t'_{j1},t_{(j+1)1},\ldots,t_{n1})].d^*,$$

where $d^*=(t_{j1}-t'_{j1})^{-1}$; otherwise, for $t'_{j1}=t_{j1}$, $p_{1j}=1$. Now rewrite (5) as

$$
\begin{aligned}
0&=[P_1(t_{11},\ldots,t_{n1})-P_1(t'_{11},\ldots,t'_{n1})]\\
&=[P_1(t_{11},\ldots,t_{n1})-P_1(t_{11},\ldots,t_{(n-1)1},t'_{n1})]\\
&\quad +[P_1(t_{11},\ldots,t_{(n-1)1},t'_{n1})-P_1(t_{11},\ldots,t'_{(n-1)1},t'_{n1})]\\
&\quad +\cdots+[P_1(t_{11},t'_{21},\ldots,t'_{n1})-P_1(t'_{11},\ldots,t'_{n1})].
\end{aligned}
$$

By WPD, $p_{1j}\geq 0$. If we define $\delta x=x'-x$, then $t'_{j1}-t_{j1}=\delta x_j+\delta x_1$ for all $j\neq 1$ and $t'_{11}-t_{11}=\delta x_1$. Using the definition of $p_{1j}$, (5) may be rewritten as

$$0=p_{11}\delta x_1+p_{12}(\delta x_2+\delta x_1)+\cdots+p_{1n}(\delta x_n+\delta x_1). \tag{5'}$$

Similarly $p_{ij}$'s for all $i,j=1,2,\ldots,n$, may be defined, which following the construction of (5′) from (5) gives

$$
\begin{aligned}
0&=p_{11}\delta x_1+p_{12}(\delta x_2+\delta x_1)+\cdots+p_{1n}(\delta x_n+\delta x_1),\\
0&=p_{21}(\delta x_1+\delta x_2)+p_{22}\delta x_2+\cdots+p_{2n}(\delta x_n+\delta x_2),\\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
0&=p_{n1}(\delta x_1+\delta x_n)+\cdots+p_{n,n-1}(\delta x_{n-1}+\delta x_n)+p_{nn}\delta x_n.
\end{aligned}
\tag{6}
$$

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Alternatively, we can write (6) in vector matrix form as

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} p_{1j} & p_{12} & \cdots & p_{1n} \\ p_{21} & \sum_{j=1}^{n} p_{2j} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & \sum_{j=1}^{n} p_{nj} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix}. \tag{6'}$$

Note that the matrix of the right-hand side would be equivalent to the Jacobian of the mappings $y_i = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$, whenever for each $i$, $f_i$ is nondecreasing and differentiable with respect to all its arguments (i.e., $\delta f_i/\delta x_j \geqq 0$ for all $i$ and $j$), since for $i, j = 1, \ldots, n$, $(\delta f_i/\delta x_j) = p_{ij}$ for $i \neq j$ and $[(\partial f_i/\partial x_i) - \sum_{j \neq i}(\delta f_i/\delta x_j)] = p_{ii}$.

*Proof of Theorem 1*

To the contrary, suppose for some assigned value of $y$, the solution vector $x$ exists and is not unique. Assume, $x$ and $x'$ are two solution vectors. Construct $p_{ij}$'s in the way explained earlier. By definition, (6) is true. We shall complete our proof by demonstrating that, $\delta x_1 = \delta x_2 = \cdots = \delta x_n = 0$.

Suppose $\delta x_i \neq 0$ for some $i$. Without any loss of generality assume $\delta x_i > 0$. We shall show that there must exist some $\delta x_j$ such that $|\delta x_j| > |\delta x_i|$. Consider the $i$th equation of system (6):

$$0 = p_{i1}(\delta x_1 + \delta x_i) + \cdots + p_{ii}\delta x_i + \cdots + p_{in}(\delta x_n + \delta x_i). \tag{7}$$

If $p_{ii} > 0$, $p_{ii}\delta x_i > 0$. By WPD, $p_{ij} \geqq 0$. Hence there exists a negative $\delta x_j$ such that $|\delta x_j| > |\delta x_i|$. If $p_{ii} = 0$, by WPD, there exists an element $a$ in the index set such that $P_{ia}$ is an increasing function of $t_{ai}$ which also implies that $P_{ia}$ is an increasing function of $x_a$. If $\delta x_i \neq \delta x_a$ then $p_{ia} > 0$. Suppose $\delta x_i < -\delta x_a$. Obviously our search for a $\delta x_j$, such that $|\delta x_j| > |\delta x_i|$, ends with the choice of $j = a$. Again if $\delta x_i > -\delta x_a$, since $p_{ii}\ \delta x_i = 0$ and $p_{ia}(\delta x_a + \delta x_i) > 0$, by (7) there exists some $\delta x_j$ which is negative and $|\delta x_j| > |\delta x_i|$.

The problem arises when $p_{ii} = 0$ and $-\delta x_a = \delta x_i$. Then consider the relation (a) of (6):

$$0 = p_{a1}(\delta x_1 + \delta x_a) + \cdots + p_{aa}\delta x_a + \cdots + p_{an}(\delta x_n + \delta x_a).$$

Again, by WPD, either there exists a positive $\delta x_j$ such that $|\delta x_j| > |\delta x_a| = |\delta x_i|$ or there exists $b$ in the index set such that $\delta x_b$ is positive and $|\delta x_b| = |\delta x_a| =$

$|\delta x_i|$. In the former situation our search for a $\delta x_j$ such that $|\delta x_j|>|\delta x_i|$ ends. In the latter situation we proceed to consider the relation (b) of (6):

$$0=p_{b1}(\delta x_1+\delta x_b)+\cdots+p_{bb}\delta x_b+\cdots+p_{bn}(\delta x_n+\delta x_b).$$

By WPD this search must end. In case of worst luck we have to continue till we reach a situation where, $|\delta x_k|=|\delta x_g|=\cdots=|\delta x_a|=|\delta x_i|$. By WPD again $p_{kk}>0$. Also,

$$0=p_{k1}(\delta x_1+\delta x_k)+\cdots+p_{kk}\delta x_k+\cdots+p_{kn}(\delta x_n+\delta x_k).$$

Depending on whether $\delta x_k$ is positive or negative now there must exist a negative or positive $\delta x_j$ such that $|\delta x_j|>|\delta x_k|=\cdots=|\delta x_i|$ to satisfy the above equation.

We have arrived at the conclusion that if $\delta x_i>0$ for some $i$, there exists $\delta x_j$ such that $|\delta x_j|>|\delta x_i|$. It is clear from the nature of our argument that if $\delta x_i<0$ from some $i$, there would also exist some $\delta x_j$ with $|\delta x_j|>|\delta x_i|$. Now, suppose $\delta x=x'-x\neq 0$. Consider a $\delta x_m$ such that $|\delta x_m|\geq|\delta x_i|$ for all $i$. We have just shown above that WPD requires the existence of a $\delta x_k$ such that $|\delta x_k|>|\delta x_m|$ which contradicts the choice of $\delta x_m$. Therefore, $\delta x=x'-x=0$, which proves our theorem. Q.E.D.

*Proof of Theorem 3*

Given Theorem 2, we have to establish only the necessity part. Suppose, WND, is not satisfied in the cell $(x_0\pm k)$ where $k$ is a positive $n$-dimensional vector $(k_1,\ldots,k_n)$. Since $f(x)$ is continuous almost everywhere, pick up $x\in(x_0\pm k)$, such that $f(x)$ is continuous in the neighborhood of $x$, the chain condition breaks down. We can divide $y=f(x)$ into two sets of functions. The set $I$ contains those functions which satisfy the chain condition and the set $J$ contains those functions which do not satisfy the chain condition. Note, for $i\in J$, $P_{ii}$ and $P_{ij}$ $(i\neq j$ and $j\notin J)$ are locally invariant in the neighborhood of $x$. Therefore, in the neighborhood of $x$, we may write, $y_i=P_i(t_{ji},\ldots,t_{ni})$, $i=j,\ldots,n$, where $t_{ii}=x_i$ and $t_{ki}=x_i-x_k$. Since $P_i$ is invariant with respect to $t_{ii}$, for $i\in J$, and $t_{ki}$ is the difference of two terms, $y_i$ $(i\in J)$ remains invariant if we add a small value $\delta>0$ to each element of the solution vector. That is, if $x_i$ $(i\in J)$ is a solution vector for $y_i=f(x_j,\ldots,x_n)$, $i\in J$, then $(x_j+\delta)$ is also a solution vector. Let $x'_i$ $(i\in I)$ be such that $y_i=(x'_1,x'_2,\ldots,x'_{j-1},x_j+\delta\ldots,x_n+\delta)$ for $i\in J$. Remember, $\delta$ is an arbitrary but small positive number and $y_i$ is a continuous function in the neighborhood of $x$ satisfying WND with respect to the variables $(x_1,\ldots,x_{j-1})$. Hence, the solution $x'_i$ $(i=1,\ldots,j-1)$ exists and satisfies a vector-matrix relationship similar to eq. (6′) in section 3. It is clear that if $x=(x_1,\ldots,x_n)$ is a solution vector then $x'=(x'_1,\ldots,x'_{j-1},$ $x_j+\delta,\ldots,x_n+\delta)$ is also a solution vector for $y=f(x)$. Q.E.D.

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