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A Note on Complementarity Over Time

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The role of complementary consumption goods in static general equilibrium analysis has been discussed by various authors. There are two widely known definitions of complementarity. The first definition is due to Pareto and Edgeworth who define x and y to be complementary consumption goods if $U_{xy}>0$. Complementarity is a symmetric relationship. This requirement is satisfied by the Edgeworth-Pareto definition because of the symmetry of cross partial derivatives. Since Edgeworth-Pareto type complementarity (henceforth called E-P complementarity) depends crucially on the choice of the utility index, a static theory of demand constructed in the framework of ordinal utility requires an alternative definition of complementarity. Hicks provided us with an alternative definition of complementarity using the substitution term (x_{rs}) in the Slutsky equation. In the Hicksian definition, x and y are complementary consumption goods if $x_{rs}>0$. Obviously, Hicksian complementarity satisfies the symmetry requirement. Note, the Hicksian criterion of complementarity is more restricted in its applicability than the Edgeworth-Pareto definition. The Hicksian criterion requires that the consumption bundle in question must be an utility maximizing bundle at some price-income situation. If the utility function is not quasi-concave, the feasible consumption set may include a consumption bundle which does not maximize utility at any price-income situation. In the context of planning over time, the Hicksian approach is analogous to the attempt to define complementarity only along the possible optimal consumption paths.

1. RYDER-HEAL COMPLEMENTARITY

Recently some writers have observed that in case of planning over time, consumption levels at different points of time may bear a relationship of complementarity with each other. The famous Wan-Brezski example runs as follows. " A person invited to a big dinner party may tend to eat a heavier breakfast and a lighter lunch." In this case one is tempted to think that for this person, consumptions at breakfast and at supper are complementary with each other. This led Ryder and Heal [2] to formulate a pseudo-Hicksian definition of complementarity over time. Before explaining their definition and its deficiencies we should describe their model briefly. Denote the current consumption level at time t as $c(t)$ and the estimated standard consumption level as $z(t)$, where

$$z(t) = \rho . e^{-\rho t} . \int_{-\infty}^{t} e^{\rho\tau} . c(\tau) . d\tau. \qquad ...(1)$$

The planner sets his objective to maximize,

$$J[c(\cdot)] = \int_{0}^{\infty} e^{-\delta t} U[c(t), z(t)]dt \qquad ...(2)$$

subject to the familiar constraints of a one-sector growth model:

$$\dot{k} = f(k) - c - nk \qquad ...(3)$$

$$0 \leqq c \leqq f(k).$$

The stationary utility function $U[c, z]$ is twice continuously differentiable and is subject to a standard set of assumptions. Along any feasible consumption path $c(\cdot)$, the marginal

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utility of consumption at t_1 is the Volterra derivative of $J[c(\cdot)]$ at t_1 and is denoted by $J'[c(\cdot), t_1]$. The marginal rate of substitution at two different dates t_1 and t_2 is given by the ratio,

$$R[c(\cdot);\ t_1, t_2] = J'[c(\cdot), t_1]/J'[c(\cdot), t_2] \qquad ...(4)$$

Let $R'[c(\cdot);\ t_1, t_2;\ t_3]$ denote the Volterra derivative of $R[c(\cdot);\ t_1, t_2]$ with respect to t_3. According to Ryder and Heal $c(t_1)$ and $c(t_3)$ are complementary (henceforth called R-H. complementary) if $R'[c(\cdot);\ t_1, t_2;\ t_3]>0$. In this case $c(t_2)$ and $c(t_3)$ may also be defined as substitutes.[1] Note that, unlike Hicksian complementarity, an increase in $c(t_3)$ has not been compensated. This is perhaps one of the reasons why (as we shall presently see) quite a few undesirable characteristics are associated with R-H complementarity.

First, the R-H complementarity is not a symmetric relationship. Even in the simplest case of a stationary consumption path, it is possible that $R'[c(\cdot);\ t_1, t_2;\ t_3]>0$ but for a change around $c(t_1)$ the rate of substitution between $c(t_2)$ and $c(t_3)$ moves against $c(t_3)$. If for any economic reason we are led to believe that $c(t_1)$ is complementary with $c(t_3)$ then for the same reason we should call $c(t_3)$ a substitute for $c(t_1)$. This, I think, is a very embarrassing possibility. However, the most objectionable feature of R-H complementarity stems from the fact that whether $c(t_1)$ and $c(t_3)$ are complementary or not depends on the choice of t_2. This is true even in a world of stationary consumption. In the model proposed by Ryder and Heal $c(t_1)$ is complementary with $c(t_3)$ along a stationary consumption path if $t_3<\alpha\cdot t_1+(1-\alpha)t_2$ where $\alpha = (\rho+\delta)/(2\rho+\delta)$. Otherwise they are substitutes. Obviously, one can choose two different values of t_2 for which the sign of $R'[c(\cdot);\ t_1, t_2;\ t_3]$ will be different.

In other words, whether breakfast is complementary to supper or not depends upon whether we are looking at the change in the rate of substitution between breakfast and lunch or the same between breakfast and high-tea. On the ground of this defect alone R-H complementarity becomes a totally meaningless concept.

2. THE EXTENSION OF EDGEWORTH-PARETO CRITERION

It seems that a proper extension of the Hicksian definition of complementarity in problems involving decision over time is a difficult task. But one can easily extend the Edgeworth-Pareto criterion for complementarity. Let $J''[c(\cdot);\ t_1, t_2]$ denote the Volterra derivative of $J'[c(\cdot), t_1]$ with respect to t_2. Using the Edgeworth-Pareto criterion for complementarity, one can say that $c(t_1)$ is complementary with $c(t_2)$ if $J''[c(\cdot);\ t_1, t_2]>0$. Accordingly, we shall call them substitutes if $J''[c(\cdot);\ t_1, t_2]<0$. Note, under quite general conditions,[2] the cross-partial Volterra derivatives are symmetric. Therefore, if $c(t_1)$ is complementary with $c(t_3)$ in the Edgeworth-Pareto sense, then usually $c(t_3)$ is complementary with $c(t_1)$. Also Edgeworth-Pareto complementarity does not require any consideration for a numeraire and consequently the problem of its being sensitive to the choice of the numeraire does not arise at all.

We shall now consider the implications of E-P complementarity in the model proposed by Ryder and Heal. In their model

$$J''[c(.);\ t_1, t_2] = \rho.\exp[\rho t_1-(\rho+\delta)t_2].U_{cz}[c(t_2), z(t_2)]$$
$$+\rho^2.\exp[\rho(t_1+t_2)].\int_{t_2}^{\infty}\{\exp[-(2\rho+\delta)t].U_{zz}(c(t), z(t))\}dt. \qquad ...(5)$$

Since, from the assumptions on the utility function, it follows that $U_{zz}\leqq 0$, the second term in the RHS of equation (5) is non-positive.[3] Therefore, the cross-partial derivative of the utility function $U(c, z)$ (which is an index for the static E-P complementarity between c and z) has an important role in determining the sign of $J''[c(\cdot);\ t_1, t_2]$. If $U_{cz}<0$, then $c(t_1)$ and $c(t_2)$ are substitutes for any arbitrary choice of t_1 and t_2. On the other hand, if $U_{cz}>0$, the sign of $J''[c(\cdot);\ t_1, t_2]$ is indeterminate. The general condition under which $c(t_1)$ and

$c(t_2)$ are complementary is too complicated and hardly admits of any economic interpretation. However, along a stationary consumption path,

$$J''[c(.);\ t_1, t_2] = \rho.\exp[\rho t_1 - (\rho+\delta)t_2].\left(U_{cz} + \frac{\rho}{2\rho+\delta}U_{zz}\right). \qquad ...(6)$$

It follows at once that if $U_{cz} > [\rho/(2\rho+\delta)]\cdot|U_{zz}|$ on any stationary path then $c(t_1)$ and $c(t_2)$ are complementary for any t_1 and t_2. Similarly, if $U_{cz} < [\rho/2\rho+\delta)]\cdot|U_{zz}|$ on any stationary path then $c(t_1)$ and $c(t_2)$ are substitutes. On any stationary consumption path whether consumptions at two different dates are complementary or not depends on the sign and the magnitude of U_{zz}/U_{cz}. *Note, that along the stationary consumption path, whether consumptions at two different dates are complementary or not does not depend on the length of the time elapsed between these two dates.* When consumptions vary over time, complementarity between $c(t_1)$ and $c(t_2)$ depends on the specific values of t_1 and t_2. Since $d(\log J''[c(\cdot);\ t_1, t_2])/dt_1$ equals ρ, if $c(t_1)$ and $c(t_2)$ are complementary for some $t_1 < t_2$ then they are so for any $t_1 < t_2$. Now consider any triplet (t_1, t_2, t_3) where $t_1 < t_2$ and both $c(t_1)$ and $c(t_2)$ are complementary with $c(t_3)$. A variation in $c(t_3)$ may lead to a substitution of $c(t_1)$ for $c(t_2)$. But this does not mean that $c(t_2)$ and $c(t_3)$ are substitutes (as in Ryder and Heal). It merely implies that the relative rate of increase in the marginal utility of $c(t_1)$ is greater than that of $c(t_2)$.

If $c(t_1)$ and $c(t_2)$ are complementary, a different choice of the unit for the measurement of utility should not affect that relationship. In other words, the sign of $J''[c(\cdot);\ t_1, t_2]$ should be invariant to any linear transformation of the utility function. Equation (5) tells us that it is so. The sign of $J''[c(\cdot);\ t_1, t_2]$ is not and need not be invariant under any monotonic transformation of the stationary utility function. After all, when considering a problem of optimization over time, one has to be a cardinalist. The optimal paths are not invariant under any monotonic transformation of the utility function.

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NOTES

1. Although the definition of " substitutes " does not appear in Heal-Ryder, but such a definition seems to be an inevitable consequence of their definition of complementarity between $c(t_1)$ and $c(t_3)$.

2. The requirement is that $J''[c(\cdot);\ t_1, t_2]$ must be a continuous functional in $c(\cdot)$ of order 0 and a continuous function of t_1 and t_2. This is a generalization of the result $f_{xy} = f_{yx}$ in the function space. It was first proved by Volterra [5]. For a reference to this result see [4, p. 24].

3. The reader should be warned that by interchanging t_1 and t_2 in the expression for $J''[c(\cdot);\ t_1, t_2]$ one does not obtain $J''[c(\cdot);\ t_2, t_1]$. They are different. The reader can work out the Volterra cross-partial derivatives from the definition and see that they are identical.

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