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Changing the Discount Rate by Adjusting the Pure Rate of Time Preference

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The Ramsey (1928) equation decomposes the real discount rate into the pure rate of time preference plus a term that accounts for the changing marginal utility of consumption. Discussions about the appropriate discount rate to apply in Cost Benefit Analysis sometimes refer to variations induced by alternative values of the pure rate of time preference as if the two vary on a one-to-one basis. But the optimal consumption path, which determines the marginal product of capital and hence the discount rate, depends on the rate of time preference. Hence the discount rate depends on time preference through the marginal utility term. We derive an analytical expression of this relationship and show that the derivative of the discount rate with respect to time preference only equals unity in the steady state and converges from below. We estimate the derivative using US data from 1930 to 2015. Based on a semi-parametric regression model with time-varying coefficients we find it is about 0.9, but we cannot rule out 1.0 being included in the 95% confidence interval. The implied pure rate of time preference after 1980 is about 1.6 percent.

Keywords: Discounting; Time Preference; Regression with Time-varying Parameters

JEL Classifications: H43, D61

1 Introduction

The Ramsey (1928) equation decomposes the discount rate *r* at time *t* into the expression:

$$r(t) = \rho + \eta \frac{\dot{C}(t)}{C(t)} \tag{1}$$

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where ρ is the pure rate of time preference, η is the consumption elasticity of marginal utility and $\dot{C}(t)/C(t)$ is the growth rate of consumption. It has been common practice in Cost-Benefit Analysis (CBA) to construct alternate values of $r(t)$ by varying ρ and $r(t)$ identically on the assumption that $\partial r(t)/\partial\rho = 1$. For example, Stern (2006) discusses the implications for discount rates in climate policy analysis from adopting different stances on the ethics of discounting the utility of future generations, Chichilnisky et al. (2018) review key debates over whether positive values of ρ are ethically defensible and Yang (2003) discusses the possibility of implementing "dual discount rates" where ρ differs between public and private investment goods. In all such cases the discussion assumes that if ρ is increased or decreased then r increases or decreases by the same amount. However, the consumption growth path is determined by maximizing the utility stream discounted by ρ, so $\dot{C}(t)$ must be a function of the rate of time preference. Hence it cannot be assumed that $\partial r(t)/\partial\rho$ equals unity. This point does not appear to have been discussed previously.

The first aim of this paper is to characterize the way that variations in ρ may affect $r(t)$ both directly and indirectly via the growth rate. We will show that while $\partial r(t)/\partial\rho = 1$ does not necessarily hold, in the context of a closed-form growth model a convergence result $\frac{\partial r(t)}{\partial\rho} \rightarrow 1$ holds as $t \rightarrow \infty$. Note that t is not the discounting horizon but the "calendar" year, which in the context of a growth model can be understood as a measure of the proximity between the current and steady state capital stocks. We then estimate time-varying values of the coefficients in (1) on long term observations of US real interest rates and consumption growth rates and we use these to estimate $\partial r(t)/\partial\rho$. We find that the derivative is typically less than 1.0, but whether the difference is significant or not is sensitive to the exclusion of outliers associated with a period of very high real interest rates in the 1980s.

There is a large literature on the subject of choosing the appropriate discount rate for CBA (important reviews are found in Groom et al. 2005 and Arrow et al. 2014). Much attention has focused on dealing with uncertainty over long time frames (see Newell and Pizer 2003), especially as regards the growth rate of consumption. A typical formulation assumes a constant value of η and a stochastic consumption path, yielding the result that $r(t)$ should either be reduced by a constant amount or should decline as the discount horizon increases, depending on the nature of the uncertainty. However these studies take ρ as given and assume that variations in the consumption path are due to external growth shocks. Our focus here is on the question of how variations in ρ may change the consumption path even in the absence of uncertainty.

The next section decomposes the relationship between ρ and $r(t)$, first in a general formulation then using a closed-form growth model. The third section presents empirical results on US data, and the fourth section briefly concludes.

2 Endogenous Consumption Growth Rate

2.1 General Results

Differentiate equation (1) with respect to ρ:

$$\frac{\partial r(t)}{\partial \rho} = 1 + \frac{\partial \eta}{\partial \rho}\frac{\dot{C}(t)}{C(t)} + \eta(t)\frac{\partial\left(\frac{\dot{C}(t)}{C(t)}\right)}{\partial \rho} \tag{2}$$

Dropping time arguments for clarity and assuming η is constant, this reduces to:

$$\frac{\partial r(t)}{\partial \rho} = 1 + \frac{\eta}{\rho}\, g[\varepsilon_{\dot{C}} - \varepsilon_C]$$

where g is $\dot{C}/C$, namely the growth rate of consumption, ε_C is the elasticity of consumption C with respect to ρ and $\varepsilon_{\dot{C}}$ is the elasticity of the growth in consumption ($\dot{C}$) with respect to ρ. Note the derivative is undefined at $\rho = 0$, implying that the discount rate becomes highly unstable at the value most associated with intergenerational equity. If all variables are invariant to ρ, or if ρ itself is invariant, equation (2) reduces to $\frac{\partial r(t)}{\partial \rho} = 1$. To characterize the case in which ρ can vary requires some further structure.

2.2 Solution to a Closed Form Model

We solve a growth model originally due to Pezzey and Withagen (1998), as later modified by Hu and McKitrick (2013). The production function is $Y = K(t)^{\alpha}$ where Y is output, K is capital and $\alpha \in [0,1]$. The instantaneous utility function is $U = C(t)^{1-\eta}/(1-\eta)$ where η denotes the elasticity of marginal utility with respect to consumption and is positive as long as U displays risk-aversion. Capital evolves according to

$$\dot{K}(t) = K(t)^{\alpha} - C(t) \tag{3}$$

so there is no depreciation. The planner solves $\max_{C(t)} \int_{\alpha}^{\infty} e^{-\rho t} U\big(C(t)\big)\partial t$ subject to (3). The initial capital stock is denoted K_0 and the transversality condition is $\lim_{t\to\infty} e^{-\rho t} C(t)^{-1/\eta} K(t) = 0$. The usual derivation of the Ramsey equation yields

$$\alpha K(t)^{\alpha-1} = \rho + \eta\frac{\dot{C}(t)}{C(t)} \tag{4}$$

Using equations (3) and (4) we can solve for the steady state $\dot{C} = \dot{K} = 0$, obtaining the steady state capital stock and consumption levels:

$$K_{ss} = \left(\frac{\rho}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{5a}$$

$$C_{ss} = \frac{\rho}{\alpha} K_{ss} \tag{5b}$$

In order to obtain a tractable expression for the optimal capital and consumption paths it is necessary to set $\eta = 1/\alpha$ (Pezzey and Withagen 1998). The above expressions then yield the time-paths

$$K(t) = \left[K_{ss}^{1-\alpha} + (K_0^{1-\alpha} - K_{ss}^{1-\alpha})e^{-\frac{(1-\alpha)\rho t}{\alpha}}\right]^{\frac{1}{1-\alpha}} \tag{6}$$

and

$$C(t) = \frac{\rho}{\alpha} K(t) \tag{7}$$

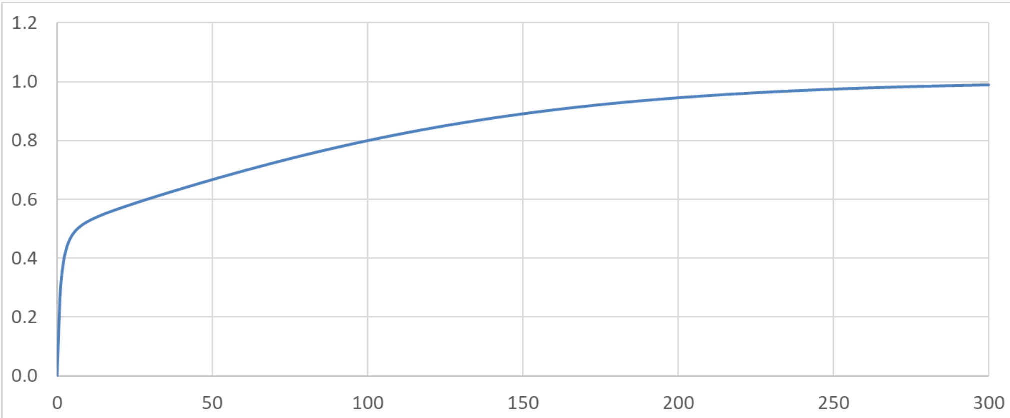
The marginal product of capital, $r(t)$, is $\alpha K(t)^{\alpha-1}$. We show in the Appendix that differentiating this with respect to ρ after applying equation (6) yields:

$$\frac{dr(t)}{d\rho} = \frac{\frac{\alpha^2}{\rho^2}(1-\theta_t)+\tilde{K}\theta_t(1-\alpha)t}{K(t)^{2-2\alpha}} \tag{8}$$

where $\tilde{K} \equiv \left(K_0^{1-\alpha} - \frac{\alpha}{\rho}\right)$ and $\theta_t = e^{\frac{(\alpha-1)\rho t}{\alpha}}$. Note that $\theta_t \to 1$ as $t \to 0$ and $\theta_t \to 0$ as $t \to \infty$. Equation (8) thus implies that $\frac{dr(t)}{d\rho} = 0$ when $t = 0$. Note also that $K(t)^{1-\alpha} = \frac{\alpha}{\rho} + \tilde{K}\theta_t$ so the limit of $K(t)^{2-2\alpha}$ as $t \to \infty$ is $\left(\frac{\alpha}{\rho}\right)^2$ and $\lim_{t\to\infty} \frac{dr(t)}{d\rho} = 1$ with the convergence from below. Hence when ρ is allowed to vary endogenously, $\frac{dr(t)}{d\rho}$ only equals unity as capital converges on the steady state value.

Figure 1 shows the profile of $dr/d\rho$ against time using equation (8) with parameter values $\alpha = 0.5$, $\rho = 0.02$ and $K_0 = 1$. From equation (5a) these parameter values imply $K_{SS} = 625$. The function begins at zero at $t = 0$ and rises quickly to exceed 0.5, then "brakes" while remaining continuously differentiable, approaching unity monotonically from below. Reducing ρ slows the rate of convergence to unity but the shape of the function remains approximately the same, as it does for any other combination of parameters that yield $K_0 < K_{SS}$.

Figure 1. Path of $\partial r/\partial \rho$ when $\alpha = 0.5$, $\rho = 0.02$ and $K_0 = 1$.



3 Estimation of $dr/d\rho$ and the pure rate of time preference

3.1 Data

Our discount rate measure was computed using two series, both obtained from the U.S. Federal Reserve. The Long Term Government Securities series (LTGOVBTD)[1] provides the earliest-available monthly average rate of return on all outstanding US government securities of 10 years or more to maturity. It is available for complete years from 1925 to 1999, of which the years from 1930 onwards were used. It was extended from 2000 to 2015 using the series IRLTLT01USM156N[2] which provides the monthly average rate of return on 10 year US Government Bonds from 1960 to the present. During the overlap period the two series exhibit a 98.4 percent correlation. They were averaged up to annual rates then converted to real rates by factoring out the annual inflation rate computed using the all-item (unchained) Consumer Price Index for Urban Consumers.[3] Annual US real personal consumption expenditures from 1930 to 2015 were also obtained from the U.S. Federal Reserve (series PCECCA)[4] and were used with annual US population to convert to per capita terms from which we computed the annual rate of change in $C(t)$. Denote this as $g(t) \equiv \dot{C}(t)/C(t)$. The resulting series are shown in Figure 2.

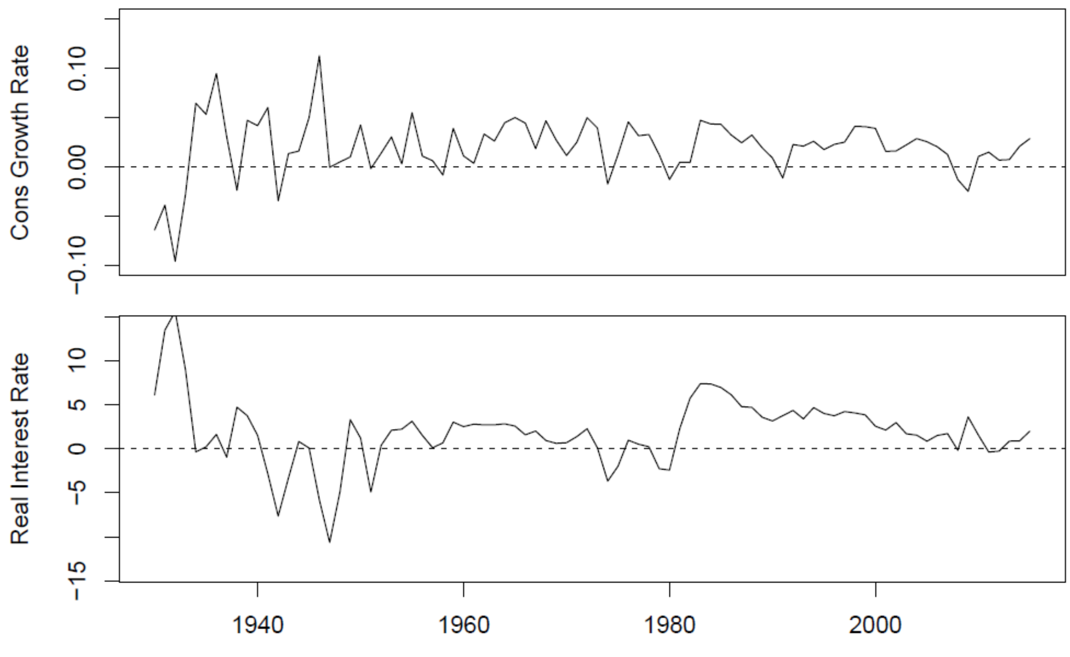
[1] Source: https://fred.stlouisfed.org/series/LTGOVTBD.

[2] Source: https://fred.stlouisfed.org/series/IRLTLT01USM156N

[3] Source: https://fred.stlouisfed.org/series/CPIAUCNS. Chained US CPI measures are not available prior to the 1960s.

[4] Source; https://fred.stlouisfed.org/series/PCECCA; population from http://www.multpl.com/united-states-population/table.

Figure 2**:** Annual percent change in US real per capita consumption expenditures (top) and real rate of return on long term US government securities (bottom) annually from 1930 to 2015.



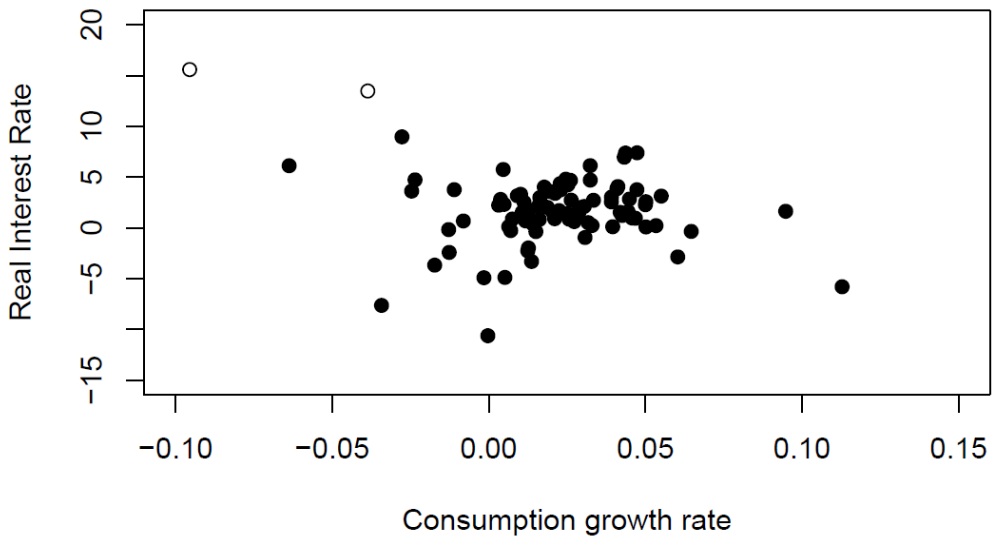
3.2 Semi-Parametric Model

Equation (1) can be implemented as a regression equation using ρ and η as parameters. A simple regression model corresponding to equation (1) would take the form

$$r(t) = \rho + \eta g(t) + e(t) \tag{9}$$

where $e(t)$ is an additive error term. OLS estimation of equation (9) yields $\hat{\rho} = 2.49$ (standard error 0.479) and $\hat{\eta} = -29.40$ (standard error 13.41). However as can be seen in Figure 3, which shows the same data as in Figure 2 but in scatterplot format, there are two potentially influential observations (shown as open circles) where the real interest rate exceeds 10 percent. Dropping just these two data points changes the OLS results considerably, to $\hat{\rho} = 1.53$ (standard error 0.465) and $\hat{\eta} = 2.53$ (standard error 13.53). The R^2 is zero and a Breusch-Godfrey test for no serial correlation strongly rejects. However OLS is not a suitable method in this case because we have established on theoretical grounds that the model coefficients are not constant over time in a growing economy so the regression model must likewise allow the intercept and slope coefficients to vary.

Figure 3: Same data as in Figure 2, shown as a scatter plot.



We do this using a Time-Varying Semi-Parametric Smooth Coefficient (TVSSC) estimator based on Robinson (1989, 1991) that allows the parameters to vary smoothly as functions of other exogenous variables including time. Our exogenous variable is the normalized time trend t/T, where T is the sample size, as recommended by Robinson (1989). This yields the semiparametric smooth coefficient estimator of the form:

$$r(t) = X(t)'\delta\left(\frac{t}{T}\right) + \epsilon(t), t = 1,2,\ldots,T \tag{10}$$

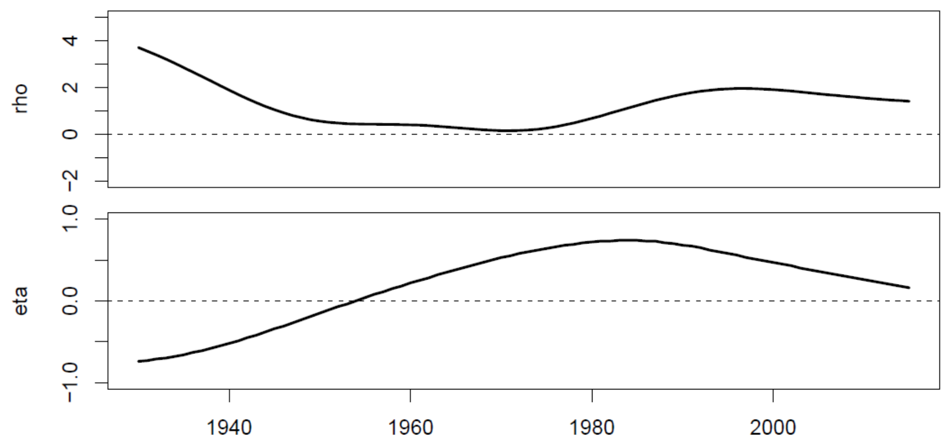
where T is the sample size, $X(t) = [1, \frac{\dot{C}(t)}{C(t)}]$ is row t from a $T \times 2$ matrix of explanatory variables where the first entry is a column of ones and the second is the vector of observations on the growth rate, δ is a pair of smooth functions of t/T corresponding to the columns of X and $\epsilon(t)$ is the error term. Denoting $z \equiv t/T$, we estimate $\delta(\cdot)$ using a local least squares estimator of the following form:

$$\hat{\delta}(z) = \left[\left(\frac{1}{Th}\right)\sum_{j=1}^{T} X(j)X'(j)K\left(\frac{z(j)-z}{h}\right)\right]^{-1} \times \left[\left(\frac{1}{Th}\right)\sum_{j=1}^{T} X(j)r(j)K\left(\frac{z(j)-z}{h}\right)\right]$$

where $K(\cdot)$ is the Gaussian kernel function, h is the smoothing parameter for sample size T, and j is the time index. Li et al. (2002) and Ozturk and Stengos (2014) provide extensive details on the properties of this estimator.

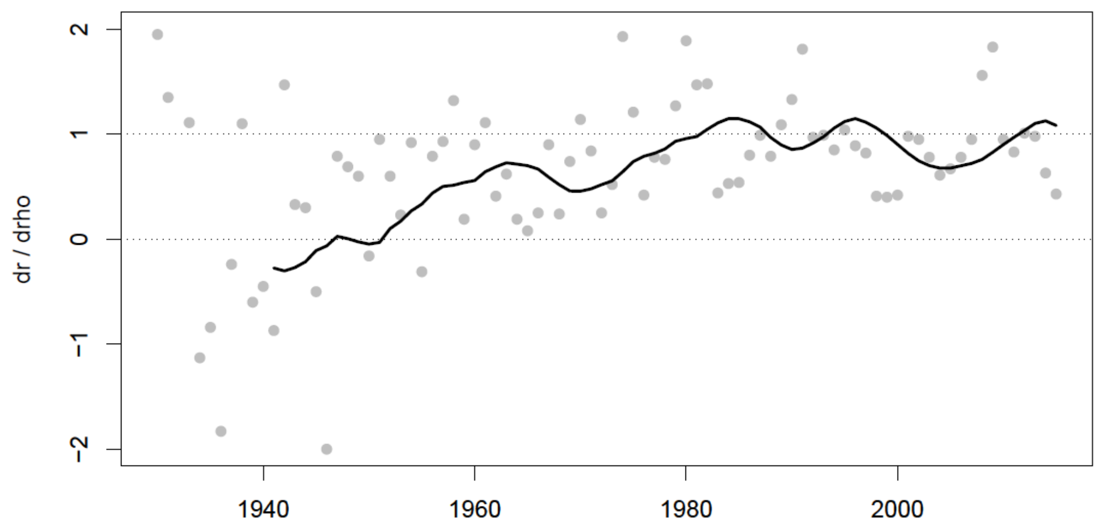
The coefficient estimates are shown in Figure 4. The estimates of $\hat{\rho}$, representing the pure rate of time preference, are highest around 1940 and decline to near zero in the 1950s and 1960s before rising to a post-1980 mean value of about 1.6 percent. Likewise our estimate of $\hat{\eta}(t)$ is not constant, begins below zero, peaks around 1980 and then reaches a small positive value by the end of the sample. A negative value of η is not consistent with risk-aversion in the utility function as it implies increasing marginal utility as consumption grows, so the estimates in the earliest decades of the sample should be considered less reliable.

Figure 4. Time-varying estimates of $\hat{\rho}$ and $\hat{\eta}$ from the TVSSC model.



We estimated $dr(t)/d\rho$ by regressing $\hat{\eta}(t)$ on $\hat{\rho}(t)$ (without an intercept), obtaining the slope coefficient, then regressing $g(t)$ on $\hat{\rho}(t)$ again without an intercept obtaining the slope coefficient, then using these coefficients in equation (2) with the TVSSC parameters. The yearly results are shown in Figure 5 (gray dots) with a 10-year moving average (applying triangular weights) post-1940 (black line). Note that some stability is imposed on this estimator by treating the derivative terms as constants. The mean value of the yearly series shown in Figure 4 is 0.67 and its standard deviation is 0.76. The time profile of the smoothed derivative is qualitatively very similar to the profile predicted by the theoretical model. While values for individual years are rather noisy and even exhibit unexpectedly negative values early in the sample, the derivative approaches unity from below as expected. The mean of the unsmoothed values in Figure 5 after 1980 is 0.91 and the standard deviation is 0.37 so a unit value for $dr/d\rho$ cannot be ruled out. While a 95 percent confidence interval does not exclude unity, neither is there any indication in Figure 5 that the smoothed line is tending towards unity, and instead seems attracted to a mean value centered around 0.9.

Figure 5. Time-varying estimates of $d\rho/dr$ from TVSSC model, 1930—2015 (gray dots) and results post-1940 smoothed using a 10-year moving average with triangular weights (black line).



4 Conclusions

Varying ρ for the purpose of computing alternate discount rates in Cost-Benefit Analysis is common practice and assumes that the two vary identically, or $dr(t)/d\rho = 1$. But the second term in the Ramsey equation (1) is also dependent on ρ so this derivative cannot be assumed to equal unity. We find on theoretical grounds that $dr(t)/d\rho < 1$ outside the steady state in an economic growth model, approaching unity from below. We find empirical support for this finding from examining long term US real interest and consumption growth. A semi-parametric model allowing for smoothly-varying coefficients over time yields an estimate in recent decades centered at around 0.9, however the confidence interval does not exclude unity. The empirical model implies a value of the pure rate of time preference after 1980 of about 1.6 percent.

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APPENDIX: Derivation of Equation (8)

From equation (6)

$$K(t) = \left[K_{SS}^{1-\alpha} + (K_0^{1-\alpha} - K_{SS}^{1-\alpha})e^{-\frac{(1-\alpha)\rho t}{\alpha}}\right]^{\frac{1}{1-\alpha}}.$$

Note $K_{ss} = \left(\frac{\rho}{\alpha}\right)^{\frac{1}{\alpha-1}} \Rightarrow K_{SS}^{1-\alpha} = \frac{\alpha}{\rho}$. Then

$$K(t)^{1-\alpha} = \frac{\alpha}{\rho} + \left(K_0^{1-\alpha} - \frac{\alpha}{\rho}\right) e^{-\frac{(1-\alpha)\rho t}{\alpha}}.$$

Denote $\widetilde{K} \equiv \left(K_0^{1-\alpha} - \frac{\alpha}{\rho}\right)$ and $\theta_t = e^{-\frac{(1-\alpha)\rho t}{\alpha}}$. Note also:

$$\frac{d\widetilde{K}}{d\rho} = \frac{\alpha}{\rho^2}$$

$$\frac{d\theta_t}{d\rho} = -\frac{\theta_t(1-\alpha)t}{\alpha}.$$

It follows that $K(t)^{1-\alpha} = \frac{\alpha}{\rho} + \widetilde{K}\theta_t$ and $r(t) = \alpha K(t)^{\alpha-1} = \alpha\left(\frac{\alpha}{\rho} + \widetilde{K}\theta_t\right)^{-1}$.

Differentiate with respect to ρ to get

$$\frac{dr}{d\rho} = -\alpha\left(\frac{\alpha}{\rho} + \widetilde{K}\theta_t\right)^{-2}\left(-\frac{\alpha}{\rho^2} + \theta_t\frac{d\widetilde{K}}{d\rho} + \widetilde{K}\frac{d\theta_t}{d\rho}\right)$$

$$= -\alpha\left(\frac{\alpha}{\rho} + \widetilde{K}\theta_t\right)^{-2} \times \left(-\frac{\alpha}{\rho^2} + \frac{\alpha}{\rho^2}\theta_t - \frac{\widetilde{K}\theta_t(1-\alpha)t}{\alpha}\right)$$

$$= \frac{\frac{\alpha^2}{\rho^2}(1-\theta_t) + \widetilde{K}\theta_t(1-\alpha)t}{K(t)^{2-2\alpha}}.$$