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On a closed-form solution of the Brock–Mirman model with a finite horizon: A shadow-value interpretation

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Abstract

This note revisits the finite-horizon Brock-Mirman model with logarithmic utility, Cobb-Douglas technology, and full depreciation. Its contribution is not a new general solution method, but a compact shadow-value representation of a canonical benchmark. Measuring current output in marginal utility units turns the Euler trade-off into a single linear recursion. This recursion yields closed-form consumption and saving shares, shows how zero terminal capital and terminal continuation values differ, and explains why finite-horizon policies converge geometrically to the infinite-horizon allocation. The same representation also clarifies the incidence of proportional wedges and multiplicative productivity shocks: in this special case, they rescale levels without changing the share recursion.

JEL codes: C61, E13, E20, O41.

Keywords: optimal growth model, finite horizon, shadow value, terminal condition, closed-form solution.

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1 INTRODUCTION

This note revisits a highly tractable version of the Brock-Mirman growth model and asks what is gained by solving its finite-horizon version through shadow values rather than through the usual sequence of backward substitutions. The answer is modest but useful. The paper does not claim to provide a new general method for optimal growth. Instead, it shows that in the canonical log-Cobb-Douglas case with full depreciation, the entire finite-horizon allocation can be organized around one shadow-value recursion. This representation makes the endpoint effect, the time profile of consumption and saving shares, and the link with the infinite-horizon benchmark transparent.

The special case is interesting for reasons beyond its tractability. It is the standard analytical benchmark used to teach dynamic programming, Euler equations, and stochastic growth. Because all shares are available in closed form, it also provides a clean diagnostic for finite-horizon truncation. A zero terminal capital condition, a terminal continuation value, and the exact infinite-horizon continuation value can be compared within the same formula. This makes the model useful as a small laboratory for understanding how terminal assumptions propagate backward through dynamic allocations.

The paper is related to several strands of the literature. The optimal-growth background begins with Ramsey (1928), Cass (1965), and Koopmans (1965). Under uncertainty, early contributions include Phelps (1962), Levhari and Srinivasan (1969), Mirman (1971), Brock and Mirman (1972, 1973), and Mirman and Zilcha (1975). Finite-horizon and recursive methods build on Bellman (1957), Blackwell (1965), and Denardo (1967), and are treated in standard references such as Stokey, Lucas, and Prescott (1989) and Ljungqvist and Sargent (2018). The argument also connects to homogeneous dynamic programming, especially Alvarez and Stokey (1998), and to envelope representations such as Benveniste and Scheinkman (1979). Because the formulas below are useful partly as truncation checks, the note also relates to numerical and computational treatments of dynamic growth problems, including Coleman (1990), Taylor and Uhlig (1990), Santos and Vigo-Aguiar (1998), Judd (1998), Santos and Peralta-Alva (2005), and Stachurski (2022). Relative to this literature, the note contributes a compact representation of a familiar benchmark, not a new theory for broad classes of economies.

The organization is as follows. Section 2 states the model. Section 3 derives the shadow-value recursion and the baseline closed form. Section 4 adds terminal continuation values and shows how zero terminal capital and the infinite-horizon benchmark appear as two boundary choices. Section 5 uses the same representation to diagnose proportional wedges. Section 6 records the stochastic multiplicative-shock extension. Section 7 concludes.

2 MODEL

Time is discrete and indexed by $t = 0, 1, \ldots, T$. Per capita consumption, capital, and output in period t are denoted by c_t, k_t, and y_t, respectively. The representative household's preference is given by $\sum_{t=0}^{T} \beta^t \ln c_t$, with $\beta \in (0, 1)$. Technology is $y_t = k_t^\alpha$, with $\alpha \in (0, 1)$, and full depreciation of capital holds. Given $k_0 > 0$, the social planner solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \ln c_t$$

subject to

$$k_{t+1} = k_t^\alpha - c_t, \qquad t = 0, 1, \ldots, T,$$

and the terminal condition $k_{T+1} = 0$. Feasible allocations have positive consumption and nonnegative next-period capital. The Euler equation is

$$\frac{c_{t+1}}{c_t} = \beta\alpha k_{t+1}^{\alpha-1}, \qquad t = 0, 1, \ldots, T-1. \tag{1}$$

3 SHADOW-VALUE RECURSION

Define the finite geometric sum

$$S_t \equiv \sum_{j=0}^{T-t} (\alpha\beta)^j = \frac{1-(\alpha\beta)^{T-t+1}}{1-\alpha\beta}. \tag{2}$$

It is the unique solution of $S_t = 1 + \alpha\beta S_{t+1}$ with $S_T = 1$.

Let s_t denote the saving share in period t, $s_t \equiv k_{t+1}/y_t$.

Proposition 1. *At each date, the planner allocates current output between consumption and next period capital using the time-varying shares* $1/S_t$ *and* $1 - 1/S_t$*. Equivalently,*

$$c_t = \frac{1}{S_t} k_t^\alpha, \qquad k_{t+1} = \left(1 - \frac{1}{S_t}\right) k_t^\alpha, \tag{3}$$

$$s_t = 1 - \frac{1}{S_t}. \tag{4}$$

Proof. Since the social planner's problem is a finite-dimensional concave optimization problem, a unique solution exists. It suffices to verify the proposed allocation. Suppose that $c_t = k_t^\alpha / S_t$. By Eq. (1),

$$\frac{S_t}{S_{t+1}} \frac{k_{t+1}^\alpha}{k_t^\alpha} = \alpha\beta k_{t+1}^{\alpha-1}.$$

Thus $k_{t+1} = \alpha\beta S_{t+1} k_t^\alpha / S_t$. The resource constraint also implies $k_{t+1} = (1 - S_t^{-1}) k_t^\alpha$. Comparing coefficients gives $S_t = 1 + \alpha\beta S_{t+1}$. The expression in Eq. (2) satisfies this recursion with $S_T = 1$, so Eq. (3) follows. The saving share in Eq. (4) is immediate. $\square$

Interpretation. At the optimum, the coefficient S_t has the shadow-value representation

$$S_t = \frac{y_t}{c_t} = \frac{k_t^\alpha}{c_t}.$$

Since $u'(c_t) = 1/c_t$ and the Lagrange multiplier on the resource constraint satisfies $\lambda_t = u'(c_t)$, we have $S_t = \lambda_t y_t$. Thus S_t is the shadow value of the current output level expressed in marginal-utility units, or equivalently the marginal-utility value of a proportional increase in output.

Consider diverting one extra unit of goods at date t from consumption to investment. The current marginal utility cost is $1/c_t = S_t/y_t$. Next period, the extra unit of capital raises output by $\alpha k_{t+1}^{\alpha-1}$. Under the policy representation $c_{t+1} = y_{t+1}/S_{t+1}$, the marginal utility gain at date $t+1$ is

$$\frac{1}{c_{t+1}}\alpha k_{t+1}^{\alpha-1} = \alpha\frac{S_{t+1}}{k_{t+1}}.$$

Equating marginal cost and discounted marginal benefit, and using the resource constraint together with $S_t = y_t/c_t$, gives the constant-coefficient recursion

$$S_t = 1 + \alpha\beta S_{t+1}.$$

The economic content is immediate. As the horizon shortens, fewer future production dates remain, so the shadow value of current output falls. Since $c_t/y_t = 1/S_t$ and $k_{t+1}/y_t = 1-1/S_t$, consumption rises as a share of output and saving falls as a share of output.

Long horizon. As $T \to \infty$ with $\alpha\beta < 1$, Eq. (2) gives $S_t \to 1/(1-\alpha\beta)$ and $s_t \to \alpha\beta$. The finite-horizon policy therefore converges to the familiar infinite-horizon Brock-Mirman allocation, $c_t = (1-\alpha\beta)k_t^\alpha$ and $k_{t+1} = \alpha\beta k_t^\alpha$. The convergence is geometric because the terminal term in Eq. (2) is geometric.

Scope of the recursion. The same calculation also shows why the result should not be read as a general solution method. For a general felicity function u, define $H_t \equiv u'(c_t)y_t$. The Euler equation implies $H_t = \alpha\beta H_{t+1}y_t/k_{t+1}$. To close this as a recursion in shares, the consumption share must be recovered from H_t alone. Log utility has exactly this property because $u'(c_t)c_t = 1$, so $H_t = y_t/c_t$ and $k_{t+1}/y_t = 1 - 1/H_t$. With isoelastic utility and coefficient different from one, $H_t = c_t^{-\sigma}y_t$ and the consumption share depends on the level of output as well as on H_t. The autonomous share recursion therefore reflects the special log-Cobb-Douglas benchmark. This limitation is useful: it identifies precisely which part of the tractability comes from the shadow-value normalization.

4 TERMINAL CONDITIONS

The zero terminal capital condition is only one possible boundary condition. The same shadow-value recursion also clarifies how a terminal continuation value affects earlier choices. Suppose that, instead of imposing $k_{T+1} = 0$, the planner can choose nonnegative terminal capital and receives a terminal value $\beta^{T+1} V_{T+1}(k_{T+1})$, where

$$V_{T+1}(k) = A + B \ln k, \qquad B \geq 0.$$

The case $B = 0$ corresponds to no continuation value and reproduces zero terminal capital. Positive B gives a value to capital left after date T.

Proposition 2. *With a logarithmic terminal continuation value, the allocation is still governed by a one-dimensional shadow-value recursion. The terminal condition changes only the boundary value of that recursion. In particular,*

$$S_T^B = 1 + \beta B, \qquad S_t^B = 1 + \alpha\beta S_{t+1}^B, \quad t = 0, 1, \ldots, T-1, \tag{5}$$

and

$$S_t^B = \sum_{j=0}^{T-t} (\alpha\beta)^j + (\alpha\beta)^{T-t} \beta B. \tag{6}$$

The corresponding allocation is $c_t = k_t^\alpha / S_t^B$ *and* $k_{t+1} = (1 - 1/S_t^B) k_t^\alpha$.

Proof. At date T, the planner chooses c_T and k_{T+1} subject to $c_T + k_{T+1} = y_T$. If $B > 0$, the first-order condition is $1/c_T = \beta B / k_{T+1}$. Since $S_T^B = y_T / c_T$ and $k_{T+1} = y_T - c_T$, this condition is equivalent to $S_T^B = 1 + \beta B$. If $B = 0$, the solution is the boundary allocation $k_{T+1} = 0$ and the same expression gives $S_T^B = 1$. For all earlier dates, the Euler equation is unchanged, so the same argument as in Proposition 1 gives $S_t^B = 1 + \alpha\beta S_{t+1}^B$. Iterating backward proves Eq. (6) and the policy functions follow from the resource constraint. $\square$

This extension is useful because it separates two issues that are often mixed in finite-horizon computations. The recursion describes the economics of intertemporal substitution inside the model. The boundary value describes what the finite problem assumes about the value of capital after the final date. If the terminal value is too low, saving is depressed near the terminal date and the effect propagates backward geometrically.

The exact infinite-horizon continuation value is a special case. In the infinite-horizon Brock-Mirman model, the coefficient on $\ln k$ in the value function is

$$B^{\infty} = \frac{\alpha}{1-\alpha\beta}.$$

Substituting $B = B^{\infty}$ into Eq. (5) gives $S_T^B = 1/(1-\alpha\beta)$, and the recursion then implies $S_t^B = 1/(1-\alpha\beta)$ for every earlier date. Thus a finite-horizon problem with the exact terminal continuation value reproduces the infinite-horizon saving share at all dates. In contrast, the zero terminal capital problem sets $B = 0$ and therefore produces a terminal saving share of zero. The difference between any terminal value and the infinite-horizon benchmark is

$$S_t^B - \frac{1}{1-\alpha\beta} = (\alpha\beta)^{T-t}\left(1 + \beta B - \frac{1}{1-\alpha\beta}\right),$$

which shows exactly how terminal-condition errors decay backward from the endpoint.

This formula gives a direct truncation diagnostic. With zero terminal capital, the relative gap in the shadow value at date t is $(\alpha\beta)^{T-t+1}$. If the terminal date is $n = T-t$ periods after the date of interest, this relative error is $(\alpha\beta)^{n+1}$. Hence the error is below ε if and only if $n+1 \geq \log(\varepsilon)/\log(\alpha\beta)$, or equivalently $n \geq \log(\varepsilon)/\log(\alpha\beta) - 1$. The closed form therefore provides a simple way to separate economic dynamics from terminal-condition artifacts in finite-horizon approximations.

5 WEDGES

The wedge extension is not intended as a substantive theory of taxation. Its role is diagnostic: it shows how the shadow-value recursion separates intertemporal shares from multiplicative wedge incidence in this benchmark case.

Consider the planner representation

$$\phi c_t + \iota k_{t+1} = \theta k_t^\alpha, \qquad \phi > 0, \quad \iota > 0, \quad \theta > 0,$$

where ϕ is a consumption-side wedge, ι is an investment-side wedge, and θ is an output-side wedge. This representation can be supported by proportional taxes and lump-sum transfers in a decentralized economy. Define the taxed-output shadow value by

$$S_t^\tau \equiv \frac{\theta y_t}{\phi c_t} = \frac{\theta k_t^\alpha}{\phi c_t}.$$

Proposition 3. *At each date, proportional wedges rescale physical allocations, while the consumption and investment shares in transformed units coincide with the baseline shares. In particular,*

$$c_t = \frac{\theta}{\phi}\frac{1}{S_t}k_t^\alpha, \qquad k_{t+1} = \frac{\theta}{\iota}\left(1 - \frac{1}{S_t}\right)k_t^\alpha, \tag{7}$$

$$\frac{k_{t+1}}{y_t} = \frac{\theta}{\iota}\left(1 - \frac{1}{S_t}\right), \qquad S_t^\tau = S_t. \tag{8}$$

Proof. The Euler condition for the wedge problem is

$$\frac{c_{t+1}}{c_t} = \beta\alpha\frac{\theta}{\iota}k_{t+1}^{\alpha-1}. \tag{9}$$

Guess $c_t = (\theta/\phi)k_t^\alpha/S_t$. Substitution into Eq. (9) gives

$$k_{t+1} = \frac{\theta}{\iota}\alpha\beta\frac{S_{t+1}}{S_t}k_t^\alpha.$$

The constraint gives $k_{t+1} = (\theta/\iota)(1 - S_t^{-1})k_t^\alpha$. Comparing coefficients again gives $S_t = 1 + \alpha\beta S_{t+1}$ with $S_T = 1$. This proves Eq. (7) and Eq. (8). □

Let $\tilde{y}_t \equiv \theta y_t$, $\tilde{c}_t \equiv \phi c_t$, and $\tilde{i}_t \equiv \iota k_{t+1}$. The constraint becomes $\tilde{c}_t + \tilde{i}_t = \tilde{y}_t$, and Proposition 3 gives $\tilde{c}_t/\tilde{y}_t = 1/S_t$ and $\tilde{i}_t/\tilde{y}_t = 1 - 1/S_t$. Thus wedges do not change the geometric share recursion in transformed units. They only map those transformed shares into physical quantities.

6 EXTENSIONS

The stochastic extension is equally simple under multiplicative productivity shocks. Let $y_t = z_t k_t^\alpha$, where z_t is strictly positive. The Euler equation is

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \alpha z_{t+1} k_{t+1}^{\alpha-1} \right].$$

Under the policy form $c_{t+1} = y_{t+1}/S_{t+1} = z_{t+1}k_{t+1}^\alpha/S_{t+1}$, the term inside the expectation simplifies pointwise to $\alpha S_{t+1}/k_{t+1}$. Hence the same recursion $S_t = 1 + \alpha\beta S_{t+1}$ and the same consumption and saving shares hold date by date. Realizations of productivity affect output and capital levels, but not the share recursion. This is not a new stochastic-growth result. It records, in the same notation, the familiar implication of logarithmic utility and Cobb-Douglas homogeneity in the Brock-Mirman environment.

7 CONCLUDING REMARKS

This note has recast the finite-horizon Brock-Mirman model in terms of a shadow-value recursion. The purpose is deliberately narrow. The paper does not claim that log-Cobb-Douglas growth is representative of broader dynamic economies, nor that the result extends mechanically beyond this special benchmark. Its contribution is to show that a familiar finite-horizon allocation can be organized around one object, the shadow value of current output in marginal-utility units.

This representation clarifies three points. First, the finite-horizon consumption and saving shares follow from a single geometric recursion. Second, terminal conditions enter only through the boundary value of that recursion, which makes the connection between zero terminal capital, terminal continuation values, and the infinite-horizon benchmark explicit. Third, proportional wedges and multiplicative shocks affect levels in this special case without changing the share recursion. These formulas therefore provide a compact analytical benchmark for teaching, checking finite-horizon truncations, and interpreting endpoint effects in dynamic growth problems.

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